PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

# A CLASS OF RIESZ-FISCHER SEQUENCES

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is proved that if $\{\lambda_n\}$ is a sequence of real numbers whose differences are nondecreasing and satisfy $\sum(\lambda_{k+1} - \lambda_k)^{-2} < \infty$, then the set of complex exponentials $\{e^{i\lambda_n x}\}$ is a Riesz-Fischer sequence in $L_2[-A, A]$ for every $A > 0$, which is to say that for any positive $A$, the equations $\int_{-A}^{A} f(x)e^{i\lambda_n x}dx = c_n$ admit a solution $f$ in $L_2[-A, A]$ for every sequence $\{c_n\}$ in $\ell_2$. In particular, if $\lambda_n = n^p$, then $\{e^{i\lambda_n x}\}$ is a Riesz-Fischer sequence when $p > \frac{1}{2}$.

## 1. INTRODUCTION

The study of nonharmonic Fourier series is concerned with completeness, series expansion, and moment properties of sets of complex exponentials $\{e^{i\lambda_n x}\}$ in various function spaces. In this note, attention is restricted to the Hilbert space $L_2$ and to the case where the $\lambda_n$ are real. Inner products in $\ell_2$ are denoted by parentheses; other Hilbert space inner products are denoted by angled brackets $\langle\cdot, \cdot\rangle$, so that in $L_2[-A, A]$, $\langle f, g\rangle = \frac{1}{2A}\int_{-A}^{A} f(x)\bar{g}(x)dx$.

**Definition.** A sequence $\{f_1, f_2, f_3, \dots\}$ of elements of a Hilbert space $H$ is a *Riesz-Fischer sequence* if the system of equations $\langle f, f_n\rangle = a_n$ has at least one solution $f$ in $H$ for every sequence $\{a_n\}$ in $\ell_2$, which is to say that its moment space contains $\ell_2$.

We take as a starting point the fundamental characterization provided by a theorem of Boas [1], as stated in Young [5, p. 155]:

**Theorem.** *Let $S = \{f_1, f_2, f_3, \dots\}$ be a sequence of vectors belonging to a Hilbert space $H$. Then $S$ is a Riesz-Fischer sequence with bound $m > 0$ if and only if the inequality*

$$m\sum |a_n|^2 \le \|\sum a_n f_n\|^2 \tag{1}$$

*holds for every finite sequence of scalars $\{a_n\}$.*

Received by the editors June 15, 1993.
1991 *Mathematics Subject Classification*. Primary 42A80.
*Key words and phrases*. Riesz-Fischer sequence, moment problem, nonharmonic Fourier series.

## 2. Result

**Theorem.** *Let* $\{\lambda_n\}$ *be a sequence of real numbers whose differences are nondecreasing and satisfy*

$$(2) \qquad \sum \frac{1}{(\lambda_{k+1}-\lambda_k)^2} < \infty.$$

*Then the set of complex exponentials* $\{e^{i\lambda_n x}\}$ *is a Riesz-Fischer sequence in* $L_2[-A, A]$ *for every* $A > 0$; *in other words the equations* $\int_{-A}^{A} f(x)e^{i\lambda_n x}dx = c_n$ *admit a solution* $f$ *in* $L_2[-A, A]$ *for every sequence* $\{a_n\}$ *in* $\ell_2$.

*Proof*. Using $a$ to represent an $\ell_2$ sequence $\{a_1, a_2, \dots\}$ and parentheses to indicate the inner product in $\ell_2$, inequality (1) is the statement that

$$(3) \qquad \frac{(Ga, a)}{(a, a)} \geq m,$$

where the $\ell_2$ operator $G$ is the Gram matrix of the set $\{f_1, f_2, f_3, \dots\}$, whose entries are $g_{ij} = \langle f_i, f_j\rangle$. It is to be shown that the eigenvalues of finite subsections of $G$ are bounded away from zero, which will follow from two observations:

(1) $Gv = 0$ implies $v = 0$, for every $\ell_2$ sequence $v$.
(2) $G = I + M$, where $M$ is a compact operator.

The first condition was proved by Paley and Wiener [3], in their Theorem XLII. For if $Gv = 0$, then $(Gv, v) = 0$; Paley and Wiener showed that whenever $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = \infty$, which follows from the hypotheses of the current theorem, then the exponentials are weakly independent over an arbitrarily short interval: $\sum v_n e^{i\lambda_n x} = 0$ a.e. only when all the $v_n$ are zero. Alternatively, one could argue that no set of exponentials satisfying the hypotheses of the theorem could be complete (Young [5] shows that a set of exponentials meeting the hypotheses of this theorem has infinite deficiency in $L_2[-A, A]$) and invoke L. Schwarz's [4] observation that an incomplete set of exponentials must be (weakly) independent.

To verify condition (2) above, observe that the diagonal entries of $G = I+M$ are unity and the off-diagonal entries are

$$(4) \qquad g_{nm} = \frac{1}{2A}\int_{-A}^{A} e^{i\lambda_n x}e^{-i\lambda_m x}dx = \frac{\sin A(\lambda_n-\lambda_m)}{A(\lambda_n-\lambda_m)}.$$

$M$ can be shown to be compact by showing that its Schmidt norm is finite. Since $G$ is symmetric, it suffices to show that

$$(5) \qquad \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} g_{ij}^2 < \infty.$$

That sum is easily bounded above,

$$(6) \quad \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} g_{ij}^2 < \frac{1}{A^2}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{(\lambda_j-\lambda_i)^2} < \frac{1}{A^2}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\frac{1}{(\lambda_{i+1}-\lambda_i)^2(j-i)^2},$$

where $(\lambda_j-\lambda_i) \leq (\lambda_{i+1}-\lambda_i)(j-i)$ follows from the assumption that differences are nondecreasing. Letting $k = j+i$ and recalling the sum of $\{1/k^2\}$, one can conclude that

$$\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} g_{ij}^2 < \frac{\pi^2}{6A^2}\sum_{i=1}^{\infty}\frac{1}{(\lambda_{i+1}-\lambda_i)^2} < \infty, \tag{7}$$

establishing the theorem. □

## 3. Remarks

The argument above applies with minor modifications to doubly infinite sequences $n = \pm 1, \pm 2, \pm 3, \ldots$. One special case of interest is $\lambda_n = n^p$, which satisfies the hypotheses of the theorem as long as $p > \frac{1}{2}$. If $p = \frac{1}{2}$, the proof above fails, and in fact the operator $M$ does not have finite Schmidt norm.

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