BANACH ALGEBRAS IN WHICH EVERY ELEMENT IS A TOPOLOGICAL ZERO DIVISOR

S. J. BHATT AND H. V. DEDANIA

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Abstract. Every element of a complex Banach algebra \((A, \| \cdot \|)\) is a topological divisor of zero, if at least one of the following holds: (i) \(A\) is infinite dimensional and admits an orthogonal basis, (ii) \(A\) is a nonunital uniform Banach algebra in which the Silov boundary \(\partial A\) coincides with the Gelfand space \(\Delta(A)\); and (iii) \(A\) is a nonunital hermitian Banach \(*\)-algebra with continuous involution. Several algebras of analysis have this property. Examples are discussed to show that (a) neither hermiticity nor \(\partial A = \Delta(A)\) can be omitted, and that (b) in case (ii), \(\partial A = \Delta(A)\) is not a necessary condition.

Theorem. Every element of a complex Banach algebra \((A, \| \cdot \|)\) is a topological divisor of zero (TDZ), if at least one of the following holds:

(i) \(A\) is infinite dimensional and admits an orthogonal basis.

(ii) \(A\) is a nonunital uniform Banach algebra (uB-algebra) in which the Silov boundary \(\partial A\) coincides with the carrier space (the Gelfand space) \(\Delta(A)\) (in particular, \(A\) is a nonunital regular uB-algebra).

(iii) \(A\) is a nonunital hermitian Banach \(*\)-algebra with continuous involution (in particular, \(A\) is a nonunital C\(^*\)-algebra).

An element \(x\) in a Banach algebra \(A\) is a TDZ if there exists a sequence \((x_n), \|x_n\| = 1, \text{ for } n = 1, 2, \ldots,\) in \(A\) such that either \(x_nx \to 0\) or \(xx_n \to 0\). An orthogonal basis \([1, 3]\) in \(A\) is a sequence \((e_n)\) in \(A\) such that: (i) each \(x \in A\) can be expressed as \(x = \sum \alpha_n e_n\), \(\alpha_n\)'s are scalars; and (ii) \(e_m e_n = \delta_{mn} e_n\), \(\delta_{mn}\) being the Kronecker delta. If \((e_n)\) is an orthogonal basis in \(A\), then \((e_n)\) is a Schauder basis \([1]\) and \(A\) is semisimple, commutative, and nonunital \([3]\). \(A\) is a uB-algebra if \(\|x^2\| = \|x\|^2\) (\(x \in A\)). Such a Banach algebra \(A\) is commutative and semisimple \([2]\). A hermitian Banach \(*\)-algebra \([2, 5]\) is a Banach \(*\)-algebra in which each \(h = h^*\) has real spectrum. The above theorem supplements the well-known result \([5, \text{ Theorem 2.3.5, p. 57}]\) that every element in a radical Banach algebra is a (two-sided) TDZ.

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Proof of the Theorem. (i) Let \((e_n)\) be an orthogonal basis in \(A\). Let \(x \in A\), \(x = \sum \alpha_n e_n\). Since \(e_m e_n = \delta_{mn} e_n\) for all \(m, n\), it follows that for any \(k\), \(\|e_k\| = \|e_k^2\| \leq \|e_k\|^2\), and hence \(\|e_k\| \geq 1\); and \(xe_k = (\sum \alpha_n e_n) e_k = \sum \alpha_n e_n e_k = \alpha_k e_k \to 0\) as \(k \to \infty\). Letting \(f_k = e_k/\|e_k\|\), \(\|f_k\| = 1\), and \(\|xf_k\| \leq \|xe_k\| \to 0\).

(ii) Let \(x \in A\), \(\varepsilon > 0\). Since \(A\) is nonunital, \(\Delta(A)\) is a noncompact locally compact Hausdorff space, and the Gelfand transform \(\hat{x}\) of \(x\) vanishes at infinity. Hence, there exists a complex homomorphism \(f \in \Delta(A)\) such that \(|f(x)| < \varepsilon/2\). Let \(U = \{g \in \Delta(A) : |g(x) - f(x)| < \varepsilon/2\}\), a neighborhood of \(f\) in the Gelfand topology on \(\Delta(A)\). For \(g \in U\), \(|g(x)| \leq |g(x) - f(x)| + |f(x)| < \varepsilon\). Also, since \(A\) is a \(uB\)-algebra, for any \(y \in A\), \(\|y\| = \|\hat{y}\|_{\infty} = \sup\{|g(y)| : g \in \Delta(A)\}\). Since \(f \in \Delta(A) = \partial A\), \([4, \text{Corollary 9.2.2, p. 225}]\) implies that there exists \(y \in A\) such that \(\|y\| = \|\hat{y}\|_{\infty} = 1\) and, for \(g \in \Delta(A)\backslash U\), \(|g(y)| < \varepsilon/\|x\|\). It follows that \(\|xy\| = \|\hat{x}\hat{y}\|_{\infty} \leq \varepsilon\). Hence there exists a sequence \((y_n)\), \(|y_n| = 1\), in \(A\), such that \(xy_n \to 0\), and the proof of (ii) is complete. In a regular commutative Banach algebra \(A\), \(\partial A = \Delta(A)\) \([4, \text{Theorem 9.2.3, p. 227}]\).

(iii) We can take \(\|x^*\| = \|x\|\) for all \(x \in A\). Let \(h \in A\), \(h = h^*\). Then, for any \(n \in \mathbb{N}\), \(inh\) is quasiregular. Hence \([5, \text{Corollary 1.5.10, p. 25}]\) implies that \(h\) is a TDZ in \(A\). It follows from this that for any \(x \in A\), \(x^* x\) is a TDZ. We may assume that \(x^* x\) is a left TDZ. Hence by \([5, \text{Lemma 1.5.1 (ii), p. 20}]\), either \(x^*\) or \(x\) is a left TDZ. By continuity of the involution, it follows that \(x\) is a TDZ.

3

Examples. (3.1) For the unit circle \(T\), the Lebesgue space \(L^p(T)\), \(1 < p < \infty\), is a convolution Banach algebra with orthogonal basis \(e_n(z) = z^n\) \((n \in \mathbb{N})\) \([3]\). This is because, for each \(f \in L^p(T)\), the Fourier series of \(f\) converges to \(f\) in the norm of \(L^p(T)\). The Banach sequence algebras \(c_0\), \(l^p\) \((1 \leq p < \infty)\), with pointwise multiplication, have \(e_n = (\delta_{nm})_{m=1}^{\infty}\) as orthogonal basis. The Hardy space \(H^p(U)\) \((1 < p < \infty)\) on an open unit disc \(U\) is a Banach algebra with Hadamard product

\[
f * g(z) = \frac{1}{2\pi i} \int_{|w| = r} f(u)g(zu^{-1})u^{-1} du, \quad |z| < r < 1, \ z \in U.
\]

It has orthogonal basis \(e_n(z) = z^n\) \((n \in \mathbb{N})\) \([3]\). In all these algebras, every element is TDZ.

(3.2) For a locally compact nondiscrete abelian group \(G\), the convolution Banach algebra \(L^1(G)\) is a nonunital hermitian Banach *-algebra with involution \(f^* (t) = \overline{f(-t)}\). Thus every element of \(L^1(G)\) is TDZ. For \(G = T\) (the unit circle), the subspaces \(C(T)\) (continuous functions) and \(C^m(T)\) (\(C^m\)-functions), \(1 \leq m < \infty\), of \(L^1(T)\) are convolution Banach algebras with respective norms

\[
\|f\|_{\infty} = \sup\{|f(z)| : z \in T\} \quad \text{and} \quad \|f\|_m = \sup_{z \in T} \sum_{k=0}^{m} \frac{|f^k(z)|}{k!}.
\]

In fact, \(C(T)\) and each \(C^m(T)\) are ideals in \(L^1(T)\); hence, they are spectrally...
invariant in $L^1(T)$. Thus $C(T)$ and $C^m(T)$ are hermitian algebras and each of their elements is TDZ.

(3.3) Let $A$ be a normed algebra with completion $\overline{A}$. If every element of $A$ is TDZ, then so is in $\overline{A}$, because TDZs in $A$ form a closed set. Let $\mathcal{F}$ be the $*$-algebra of all finite rank operators on an infinite-dimensional Hilbert space $H$. By [5, p. 279], every element of $\mathcal{F}$ is a zero divisor in $\mathcal{F}$. Hence it follows that in the Banach algebras $(C^p(H), \| \cdot \|_p)$, $1 \leq p < \infty$, of operators of Schatten class $C^p$, every element is TDZ.

(3.4) In the above Theorem, neither the condition $\partial A = \Delta(A)$ in (ii) nor the hermiticity condition in (iii) can be omitted. Let $A = A(D)$, the supnorm disc algebra of the closed unit disc $D$, viz., the algebra of all continuous functions on $D$ that are analytic in the interior of $D$, with the involution $f^*(z) = \overline{f(z)}$ ($f \in A$). Let $I = \{f \in A: f(0) = 0\}$, a closed ideal in $A$. Note that $I$ (as well as $A(D)$) is not hermitian and $\partial I \neq \Delta(I)$. The function $f(z) = z$, $z \in D$, in $I$ is not TDZ in $I$.

(3.5) There exists a uB-algebra $A$ in which every element is TDZ but $\partial A \neq \Delta(A)$. For $1 < r < R$, define $U_R = \{z \in C: |z| < R\}$, $\overline{U}_R$ is the closure of $U_R$, $A = \{f \in C(\overline{U}_R): f$ is analytic in open unit disc$\}$, and $B = \{f \in A: f(z) = 0$, $r \leq |z| \leq R\}$. Then $B$ is a nonunital uB-algebra in which [4, Corollary 9.5.1, p. 246] implies that every element of $B$ is TDZ. But $\partial B = \{z: 1 \leq |z| < r\}$ and $\Delta(B) = \{z: |z| < r\}$.

(3.6) Let $T$ be a bijective bounded linear operator on an infinite-dimensional Hilbert space $H$. Let $C^*(T)$ = the operator norm closure of polynomials (without constant terms) in $T$ and $T^*$. It follows from (iii) of the above Theorem that the identity operator belongs to $C^*(T)$.

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DEPARTMENT OF MATHEMATICS, SARDAR PATEL UNIVERSITY, VALLABH VIDYANAGAR 388 120, GUJARAT, INDIA