DESCRIPTIONS OF CONDITIONAL EXPECTATIONS INDUCED BY NON-MEASURE-PRESERVING TRANSFORMATIONS

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Abstract. Given a measure-preserving transformation $T$ acting on a $\sigma$-finite measure space $(X, \mathcal{A}, m)$ and a $\sigma$-finite sigma algebra $\mathcal{B} \subseteq \mathcal{A}$, the conditional expectations $E(\cdot|\mathcal{B})$ acting on $L^\infty(\mathcal{A})$ and $E(\cdot|T^{-1}\mathcal{B})$ acting on $L^\infty(T^{-1}\mathcal{A})$ are known to be related by the formula $[E(f|\mathcal{B})] \circ T = E(f \circ T|T^{-1}\mathcal{B})$. In this note the conditional expectation $E(\cdot|T^{-1}\mathcal{B})$ is investigated in the non-measure-preserving case, and those transformations for which the above equation holds are characterized in terms of measurability conditions for $d(m \circ T^{-1})/dm$. It is precisely in the non-measure-preserving case that the measurability of $d(m \circ T^{-1})/dm$ plays an important role. Relatedly, it is shown that if composition by $T$ intertwines $E(\cdot|\mathcal{B})$ and any mapping $\Lambda$, then $\Lambda$ is a conditional expectation induced by a measure equivalent to $m$. These results were motivated by a result concerning induced conditional expectation operators on $C^*$-algebras, and the paper concludes with a brief description of this $C^*$-algebra setting.

PRELIMINARY REMARKS

Let $(X, \mathcal{A}, m)$ be a $\sigma$-finite space and $T: X \to X$ a measurable transformation. Throughout this paper we assume that $T^{-1}\mathcal{A} \subseteq \mathcal{A}$ and that $m \circ T \ll m$. We denote by $h$ the Radon-Nikodym derivative $d(m \circ T^{-1})/dm$, and we will assume throughout that $h \in L^\infty$. We denote by $E_{\mathcal{B}}$ the conditional expectation $E(\cdot|\mathcal{B})$ considered as a bounded linear transformation from $L^\infty(\mathcal{A})$ onto $L^\infty(\mathcal{B})$. If $\mathcal{A} \supseteq \mathcal{B} \supseteq \mathcal{C}$, then $E_{\mathcal{C}}$ denotes the appropriate conditional expectation from $L^\infty(\mathcal{B})$ onto $L^\infty(\mathcal{C})$. Note that $E_{\mathcal{B}}E_{\mathcal{A}} = E_{\mathcal{A}}$. Finally, given a sigma algebra $\mathcal{B}$, we denote by $\mathcal{B}_n$ the sigma algebra $T^{-n}\mathcal{B}$.

Unless expressly stated otherwise, all functions and sets are assumed to be, or constructed so as to be, $\mathcal{A}$-measurable. All sigma algebras encountered are assumed to be sigma finite with respect to $m$. All set and function statements are to be interpreted as holding up to $m$-null sets. This includes statements regarding the disjointness of sets. For a given measurable function $f$, we let $S_f$ be the support of $f$ so that $S_f = \{f > 0\}$. The reader will note that at no point in the following discussion are an uncountable number of supports considered, and consequently none of the attendant "measurability versus
selection” problems are encountered. Since $S_h$ will be encountered often, we give it special notational status by calling it $H$.

The following facts will be applied often in this article:

- $T^{-1}H = X$ [4].
- $f \circ T = g \circ T$ if and only if $\chi_H f = \chi_H g$ [1]. (We shall apply this fact most often in the form: $f \circ T = 0$ if and only if $hf = 0$.)
- $E^B(g) = g$ if and only if $g$ is $B$-measurable.
- If $f$ is $B$-measurable, then $E^B(fg) = fE^B(g)$.
- $|E^B(fg)|^2 \leq E^B(|f|^2) \cdot E^B(|g|^2)$ (conditional Cauchy-Schwarz inequality).
- If $f \geq 0$, then $E^B(f) \geq 0$. If $f > 0$, then $E^B(f) > 0$.
- For each nonnegative $F$-measurable function $f$, $S_f \subset S_{E^F f}$ [5, §1].
- For each $A$-measurable set $A$, $S_{E^A f}$ is the smallest $B$-measurable set containing $A$ [5, §1].

**Main results**

It is shown in [6] that if $T$ is measure preserving (i.e., $h = 1$ a.e. $dm$), then for any $B$,

$$\forall f \in L^\infty(B), \quad (E^B f) \circ T = E^B(f \circ T).$$

In particular, if $T$ is measure preserving, then the mapping $\Lambda: L^\infty(A) \to L^\infty(B)$ given by $\Lambda(f \circ T) = (E^B f) \circ T$ is well defined. Before considering the validity of (1) in situations where $T$ is not measure preserving, we address the question of the existence of the map $\Lambda$. We shall see later that this question is related to the study of $C^*$-algebra conditional expectations.

**Lemma 1.** $\Lambda$ is well defined if and only if $H \in B$.

**Proof.** If $\Lambda$ is well defined, then it is linear. Thus $\Lambda$ is well defined if and only if whenever $f \in L^\infty(A)$ and $f \circ T = 0$, $(E^B f) \circ T = 0$; equivalently, if $hf = 0$, then $hE^B f = 0$, i.e., $\chi_H f = 0 \Rightarrow \chi_H E^B f = 0$. Suppose first that $H$ is in $B$ and $\chi_H f = 0$. Then $E^B(\chi_H f) = 0$. Since $H \in B$, the required implication is established.

Now suppose that $\Lambda$ is well defined. If $H = X$, then $H$ is certainly $B$-measurable. Assuming $m(X - H) > 0$, then $\chi_H(1 - \chi_H) = 0$, so $\chi_H E^B(1 - \chi_H) = 0$. Thus $\chi_H = \chi_H E^B \chi_H$. We may then apply $E^B$ to both sides of the preceding equality and deduce that

$$E^B \chi_H = E^B(\chi_H E^B \chi_H) = (E^B \chi_H)^2.$$

Thus there is a $B$-set $H_1$ such that $E^B \chi_H = \chi_H$1. It then follows (see the preliminary remarks) that $H_1 \supset H$. But this shows that $\chi_{H_1} - \chi_H \geq 0$, while

$$E^B(\chi_{H_1} - \chi_H) = \chi_{H_1} - E^B \chi_H = \chi_{H_1} - \chi_H = 0.$$

Thus, $H = H_1$, so in particular, $H \in B$. □

**Remark.** The preceding result is summarized by the statement that the following diagram, wherein $C_T$ is the operator of composition by $T$, is commutative.
exactly when $H \in \mathcal{B}$:

$$L^\infty(\mathcal{A}) \xrightarrow{C_T} L^\infty(\mathcal{A}_1)$$

$$E^\mathcal{A} \downarrow \Lambda$$

$$L^\infty(\mathcal{B}) \xrightarrow{C_T} L^\infty(\mathcal{B}_1)$$

Later in this article we will briefly discuss $C^*$-algebra conditional expectations. This class includes all conditional expectations of the form $E^\mathcal{A}_1$ (referred to henceforth as classical conditional expectations) and (when $H \in \mathcal{B}$) the map $\Lambda: L^\infty(\mathcal{A}_1) \to L^\infty(\mathcal{B}_1)$ defined above. Since $E^\mathcal{A}_1$ is known to agree with $\Lambda$ when $T$ is measure preserving, it seems reasonable to examine the relationship between these mappings in general (with a proviso of course that $H$ be in $\mathcal{B}$).

To this end we now characterize $E^\mathcal{A}_1$ in terms of $h$ and $E^\mathcal{B}$. This characterization, presented in Proposition 3, does not assume the $\mathcal{B}$-measurability of $H$. We continue to use $H_1$ to denote the support of $E^\mathcal{B}h$ and note that since $T^{-1}H=X$, $T^{-1}H_1=X$ as well. The following observation and its corollary will be called into use several times in this paper.

**Lemma 2.** Let $f \geq 0$. Then $S_{E^\mathcal{B}f} \circ T \supset S_{f \circ T}$.

**Proof.** $S_{E^\mathcal{B}f} \supset S_f$ and consequently $T^{-1}S_{E^\mathcal{B}f} \supset T^{-1}S_f$. But for any $g$, $T^{-1}S_g = S_{g \circ T}$. □

**Corollary.** $(E^\mathcal{B}h) \circ T > 0$ a.e.

**Proposition 3.** For every $f \in L^\infty(\mathcal{A})$,

$$E^\mathcal{A}_1(f \circ T) = \frac{(E^\mathcal{B}(hf)) \circ T}{(E^\mathcal{B}h) \circ T}.$$

**Proof.** Let $f \in L^\infty(\mathcal{A})$, and let $B \in \mathcal{B}$. Since $|E^\mathcal{B}(hf)|^2 \leq (E^\mathcal{B}h^2)(E^\mathcal{B}|f|^2)$, $S_{E^\mathcal{B}hf} \subset H_1$. Thus

$$\int_{T^{-1}B} E^\mathcal{B}(f \circ T) \, dm = \int_{T^{-1}B} f \circ T \, dm = \int_B hf \, dm = \int_B E^\mathcal{B}(hf) \, dm$$

$$= \int_B \chi_{H_1} E^\mathcal{B}(hf) \, dm$$

$$= \int_{B \cap H_1} E^\mathcal{B}(hf) \, dm = \int_{B \cap H_1} \frac{E^\mathcal{B}(hf)}{E^\mathcal{B}h} \, E^\mathcal{B}h \, dm$$

$$= \int_{B \cap H_1} \frac{E^\mathcal{B}(hf)}{E^\mathcal{B}h} \, dm = \int_{T^{-1}(B \cap H_1)} \frac{(E^\mathcal{B}(hf)) \circ T}{(E^\mathcal{B}h) \circ T} \, dm$$

$$= \int_{T^{-1}B} \frac{(E^\mathcal{B}(hf)) \circ T}{(E^\mathcal{B}h) \circ T} \, dm \quad \text{(noting that } T^{-1}H_1 = X).$$

Since $T^{-1}B$ is a generic member of $\mathcal{B}_1$, the proof is complete. □

The following corollary gives necessary and sufficient conditions for the validity of (1) in the non-measure-preserving case:
Corollary. $E_{\mathcal{B}}^G(f \circ T) = (E_{\mathcal{B}}^G f) \circ T$ for all $f \in L^\infty(\mathcal{F})$ if and only if $h$ is $\mathcal{B}$-measurable.

Proof. If $h$ is $\mathcal{B}$-measurable, then by Proposition 3,

$$E_{\mathcal{B}}^G(f \circ T) = \frac{(E_{\mathcal{B}}^G (hf)) \circ T}{(E_{\mathcal{B}}^G h) \circ T} = \frac{(h \circ T)(E_{\mathcal{B}}^G f) \circ T}{h \circ T} = (E_{\mathcal{B}}^G f) \circ T.$$  

Conversely, suppose that the second and fourth expressions in the above equation are known to be equal for every bounded $f$ in $L^\infty(\mathcal{F})$. Then, in particular (taking $f = h$), $(E_{\mathcal{B}} h^2) \circ T = (E_{\mathcal{B}}^G h)^2 \circ T$. This may be restated as $h(E_{\mathcal{B}} h^2) = h(E_{\mathcal{B}}^G h)^2$. We then apply $E_{\mathcal{B}}$ to deduce that $(E_{\mathcal{B}} h)(E_{\mathcal{B}} h^2) = (E_{\mathcal{B}}^G h)^3$. But since $h$ is nonnegative, $E_{\mathcal{B}} h$ and $E_{\mathcal{B}} (h^2)$ have the same support (namely, $H_1$), so that $E_{\mathcal{B}} (h^2) = (E_{\mathcal{B}}^G h)^2$. We are now able to employ the variance type calculation:

$$0 \leq E_{\mathcal{B}} ((h - E_{\mathcal{B}} h)^2) = E_{\mathcal{B}} (h^2 - 2hE_{\mathcal{B}} h + (E_{\mathcal{B}} h)^2)$$

$$= E_{\mathcal{B}} (h^2) - 2E_{\mathcal{B}} (h)^2 + (E_{\mathcal{B}}^G h)^2 = E_{\mathcal{B}} (h^2) - (E_{\mathcal{B}}^G h)^2 = 0.$$  

But this is possible if and only if $h = E_{\mathcal{B}} h$ a.e., that is, $h$ is $\mathcal{B}$-measurable. □

Our next result shows that (when defined) $\Lambda$ is a classical conditional expectation, with respect to a measure equivalent to $m$. We will make use of the following test proved in [3, §3.4] for invariant measures for Markov operators (i.e., positive contractions) on an $L^1$ space. (In addition to the proof of this and related results, Krengel presents useful and interesting historical notes on these topics.)

Proposition 4. Let $M$ be a Markov operator on $L^1(X, \mathcal{F}, \nu)$, and let $C \in \mathcal{F}$. Then there exists a nonnegative $L^1$ function $g$ with $C \subseteq S_g$ and $M g = g$ if and only if for every nonnegative $L^\infty$ function $f$ with $\emptyset \neq S_f \subseteq C$, inf$_{n \geq 0} \int_X M^n f \, d\nu > 0$.

Proposition 5. Suppose that $H \in \mathcal{B}$. Then there exists a measure $\mu$ on $\mathcal{A}$ equivalent to $m|\mathcal{A}$ such that $\Lambda$ is the classical conditional expectation from $L^\infty(\mathcal{A}; \mu)$ onto $L^\infty(\mathcal{B}; \mu)$.

Proof. Define the mapping $L$ on $L^1(\mathcal{A}; m)$ by

$$L(g \circ T) = \frac{(E_{\mathcal{B}} (hg)) \circ T}{h \circ T}.$$  

Then

$$\int_X |L(g \circ T)| \, dm \leq \int_X \frac{(E_{\mathcal{B}}(h|g|)) \circ T}{h \circ T} \, dm = \int_X |h| h \frac{E_{\mathcal{B}}(h|g|)}{h} \, dm$$

$$= \int_H E_{\mathcal{B}}(h|g|) \, dm = \int_H h|g| \, dm = \int_X |g \circ T| \, dm,$$

showing that $L$ is a positive contraction on $L^1(\mathcal{A}; m)$. Moreover, for $f \circ T$ in $L^\infty(\mathcal{A})$ and $g \circ T$ in $L^1(\mathcal{B})$,

$$\int_X f \circ TL(g \circ T) \, dm = \int_X f \circ T \frac{(E_{\mathcal{B}} (hg)) \circ T}{h \circ T} \, dm = \int_H f E_{\mathcal{B}} (hg) \, dm$$

$$= \int_H h (E_{\mathcal{B}} f) \, g \, dm = \int_X (E_{\mathcal{B}} f) \circ T (g \circ T) \, dm.$$
This shows that $\Lambda = L^*$. We may now apply Proposition 4 with $C = X$. Let $f \circ T$ be a nonnegative element of $L^\infty(\mathcal{A}_1; m)$ which is not identically zero. Then $\chi_H f$ is not identically zero, and since $H$ is in $\mathcal{B}$, $\chi_H E^\mathcal{B} f = E^\mathcal{B} (\chi_H f)$ is not identically zero. This shows that $\Lambda (f \circ T) = (E^\mathcal{B} f) \circ T$ is nonnegative and is positive on a set of positive measure. Since for all $n \geq 1$, $\Lambda^n = \Lambda = L^*$, we see that $\inf_{n \geq 0} \int_X L^n f \, dm > 0$. Thus there is a strictly positive $L^1(\mathcal{A}_1)$ function $g \circ T$ such that $L(g \circ T) = g \circ T$. Let $d\mu = g \, dm$, and let $F$ be the classical conditional expectation from $L^\infty(\mathcal{A}_1; \mu)$ to $L^\infty(\mathcal{B}_1; \mu)$. (Since the two measures appearing in this argument are equivalent, there is no difference in their $L^\infty$ spaces. However, the conditional expectation operators are intimately related to their respective measures). Then for any $f \circ T$ in $L^\infty(\mathcal{A}_1; \mu)$ and $B \in \mathcal{B}$,

$$\int_{T^{-1}B} F(f \circ T) \, d\mu = \int_{T^{-1}B} f \circ T \, d\mu = \int_{T^{-1}B} f \circ T g \circ T \, dm$$

$$= \int_{T^{-1}B} f \circ TL(g \circ T) \, dm = \int_X \chi_{T^{-1}B} f \circ TL(g \circ T) \, dm$$

$$= \int_X \Lambda(\chi_{T^{-1}B} f \circ T)(g \circ T) \, dm$$

$$= \int_X \chi_{T^{-1}B} \Lambda(f \circ T)(g \circ T) \, dm = \int_{T^{-1}B} \Lambda(f \circ T) \, d\mu.$$

Since the integrands at both ends of this chain of equalities are $\mathcal{B}_1$ measurable and $T^{-1}B$ is an arbitrary $\mathcal{B}_1$ set, $F(f \circ T) = \Lambda(f \circ T)$. □

Remark. This proof assures us of the existence of an invariant measure for $L$ but does not describe it explicitly. In Proposition 6 we are able to construct invariant measures for $L$ directly because $H$ is $\mathcal{B}$-measurable. Although the proof we have given of Proposition 5 is nonconstructive, it should allow generalizations to other situations.

**Proposition 6.** Suppose that $H \in \mathcal{B}$.

(a) There is a function $g$ such that $g \circ T$ is strictly positive and in $L^1$, and $hg$ is $\mathcal{B}$-measurable.

(b) $L(g \circ T) = g \circ T$.

(c) If $m(H) < \infty$, then $g$ may be chosen to be $1/h \circ T$.

**Proof.** Since $\mathcal{B}$ is $\sigma$-finite and $H \in \mathcal{B}$, we may write $H$ as the union of a sequence of disjoint sets of finite measure from $\mathcal{B}$; say $H = \bigcup_{i=1}^{\infty} B_i$. Choose $\{\alpha_1, \alpha_2, \ldots\}$ to be a sequence of positive numbers so that the $\mathcal{B}$-measurable function $\sum_{i=1}^{\infty} \alpha_i \chi_{B_i}$ is in $L^1$. Define $g = (1/h) \sum_{i=1}^{\infty} \alpha_i \chi_{B_i}$. Then $hg$ is $\mathcal{B}$-measurable and

$$\int_X g \circ T \, dm = \int_X hg \, dm = \int_X \sum_{i=1}^{\infty} \alpha_i \chi_{B_i} \, dm < \infty.$$

In order to establish the validity of part (b), note that since $hg$ is $\mathcal{B}$-measurable,

$$L(g \circ T) = \frac{(E^\mathcal{B}(hg)) \circ T}{h \circ T} = \frac{(hg) \circ T}{h \circ T} = g \circ T.$$

Finally, suppose that $m(H)$ is finite. Then $\int_X (1/h \circ T) \, dm = m(H)$, and $hg = \chi_H$. □
C*-ALGEBRA CONDITIONAL EXPECTATION OPERATORS

Let $\mathcal{A}$ be a unital C*-algebra with identity element $1$, and let $\mathcal{B}$ be a C*-subalgebra of $\mathcal{A}$. A conditional expectation operator from $\mathcal{A}$ onto $\mathcal{B}$ is a mapping $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ satisfying

(i) $\Phi(b) = b$ for all $b \in \mathcal{B}$,
(ii) $\Phi(ab) = \Phi(a)b$ and $\Phi(ba) = b\Phi(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$,
(iii) $\Phi(a)$ is positive for all positive $a \in \mathcal{A}$,

e.g., a conditional expectation operator from $\mathcal{A}$ onto $\mathcal{B}$ is a positive, $\mathcal{B}$-linear projection from $\mathcal{A}$ onto $\mathcal{B}$. (Recall that if $x \in \mathcal{A}$, then $x$ is called positive if $x = x^*$ and the spectrum of $x$ lies on the nonnegative real axis or, equivalently, if $x = y^*y$ for some $y \in \mathcal{A}$.) Henceforth, the notation $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ will always imply that $\Phi(\mathcal{A}) = \mathcal{B}$. When $\mathcal{A} = L^\infty(\mathcal{M})$ and $\mathcal{B} = L^\infty(\mathcal{N})$, the mapping $E^\mathcal{B}$ is a conditional expectation operator. (Of course, many examples of conditional expectation operators between nonabelian C*-algebras exist and are of importance in the study of these algebras. A good general reference is Stratila [7].)

The following lemma is formulated and proved in [2]. We reproduce the proof here for the convenience of the reader.

Let $\mathcal{A}_1$ be a second C*-algebra, and let $\pi: \mathcal{A} \rightarrow \mathcal{A}_1$ be a *-homomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$. Then $\mathcal{B}_1 = \pi(\mathcal{B})$ is a C*-subalgebra of $\mathcal{A}_1$. If $\Phi(\ker \pi) \subset \ker \pi$, then $\Phi$ induces a mapping $\Phi_1: \mathcal{A}_1 \rightarrow \mathcal{B}_1$ defined by $\Phi_1(\pi(x)) = \pi(\Phi(x))$.

**Lemma 7.** Suppose $\mathcal{A}_1$ is a C*-algebra and $\pi: \mathcal{A} \rightarrow \mathcal{A}_1$ is a unital *-homomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$ such that $\Phi(\ker \pi) \subset \ker \pi$. Let $\mathcal{B}_1 = \pi(\mathcal{B})$. Then $\Phi_1$ is a conditional expectation operator from $\mathcal{A}_1$ onto $\mathcal{B}_1$.

**Proof.** If $x \in \mathcal{A}$, $\Phi_1(\Phi_1(\pi(x))) = \Phi_1(\pi(\Phi(x))) = \pi(\Phi(\Phi(x))) = \pi(\Phi(x)) = \Phi_1(\pi(x))$, so $\Phi_1$ is a projection. Also, $b, c \in \mathcal{B}$,

$$\Phi_1(\pi(b)\pi(x)\pi(c)) = \Phi_1(\pi(bxc)) = \pi(\Phi(bxc)) = \pi(b\Phi(x)c) = \pi(b)\pi(\Phi(x))\pi(c) = \pi((b)\Phi_1(\pi(x))\pi(c).$$

Thus, $\Phi_1$ is $\mathcal{B}_1$-linear. Finally, $\Phi_1(\pi(x)^*\pi(x)) = \Phi_1(\pi(x^*x)) = \pi(\Phi(x^*x))$. Since $\Phi$ is positive and $\pi$ is a homomorphism, we see that $\Phi_1$ is a positive map. Thus, $\Phi_1$ is a conditional expectation operator. □

Thus, the mapping $\Lambda: L^\infty(\mathcal{M}) \rightarrow L^\infty(\mathcal{N})$ given by $\Lambda(f \circ T) = (E^\mathcal{B} f) \circ T$ is a conditional expectation operator whenever it is well defined. Indeed, we can formulate the following generalization of Lemma 1 in the C*-algebra setting. It can be proved in the same manner as Lemma 1.

**Lemma 8.** Suppose $\mathcal{A}_1$ is a C*-algebra and $\pi: \mathcal{A} \rightarrow \mathcal{A}_1$ is a unital *-homomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$ such that there exists a selfadjoint idempotent $p \in \mathcal{A}$ with the property that $\pi(x) = 0$ if and only if $px = 0$. Let $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ be a conditional expectation operator. Suppose that $p$ commutes with each element of $\mathcal{B}$. Then the induced conditional expectation operator $\Phi_1: \mathcal{A}_1 \rightarrow \mathcal{B}_1$ is well defined if and only if $p \in \mathcal{B}$, thus if and only if $p$ lies in the center of $\mathcal{B}$.
References


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