PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

A REMARK ON DISTRIBUTION OF ZEROS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

ZHENG JIAN HUA

(Communicated by Albert Baernstein II)

ABSTRACT. The main purpose of this paper is to prove a sharp estimate of the order $\rho(w)$ of a transcendental solution w in the complex plane of an n th-order linear differential equation with polynomial coefficients in terms of the distribution of its Stokes rays, under the assumption that zero is not a Nevanlinna deficient value of w. If, in addition, there are only two Stokes rays and if all the solutions of the equation have order at most $\rho(w)$, then we can conclude that the coefficients of the equation are all constants.

1. INTRODUCTION AND RESULTS

For the sake of convenience, let us first introduce some notation:

$D = D(\theta_1, \theta_2, \dots, \theta_n)$ denotes a system of rays $D = \bigcup_{j=1}^n \{z;\ \arg z = \theta_j\}$, $0 \le \theta_1 < \cdots < \theta_{n+1} = \theta_1 + 2\pi$.

ω (resp. Θ) $= \omega(D)$ (resp. $\Theta(D)$) $=$ max (resp. min)$\{\frac{\pi}{\theta_{j+1}-\theta_j};\ 1 \le j \le n\}$.

$V_j(\varepsilon) = \{z; |\arg z - \theta_j| < \varepsilon\}$.

$G(D, \varepsilon) = C \setminus \bigcup_{j=1}^n V_j(\varepsilon)$.

Let $w(z)$ be a function meromorphic in the complex plane. We will say that the zeros of a meromorphic function $w(z)$ are attracted to D, provided that for any $\varepsilon > 0$,

$$n\left(r, G(D, \varepsilon), \frac{1}{w}\right) = o(T(r, w)),$$

as $r \to +\infty$, where $n(r, G(D, \varepsilon), 1/w)$ is the number of zeros of $w(z)$ lying in $G(D, \varepsilon) \cap \{|z| < r\}$. Throughout we denote by $\rho(w)$ (resp. $\lambda(w)$) the order (resp. lower order) of $w(z)$ and assume that the reader is familiar with Nevanlinna theory of meromorphic functions and the standard notation.

Discussions of the oscillation of the second-order linear differential equation (SDE)

$$w'' + A(z)w = 0, \tag{1}$$

where $A(z)$ is a polynomial, are very complete and delicate. The reader is referred to Gundersen [10], Hille [11, Chapter 5], and Bank and Laine [1]. In

Received by the editors November 2, 1992 and, in revised form, March 19, 1993 and June 22, 1993.

1991 *Mathematics Subject Classification*. Primary 30D35, 34A20.

This work was partially supported by a research grant from Tsing Hua University, Beijing.

0002-9939/95 $1.00 + $.25 per page

[10], some of the results clearly and completely describe the distribution of zeros of solutions of SDE (1).

Let $w(z)$ be a transcendental solution of SDE (1). Write $A(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, and set

$$\theta_j = \frac{2\pi(j+1) - \arg a_n}{n+2}, \qquad j = 0, 1, 2, \ldots, n+1. \tag{2}$$

Lemma 2 in [10] asserts that if $w(z)$ has infinitely many zeros in $V_j(\varepsilon)$ for some j, then the number of those zeros is

$$n\left(r, V_j(\varepsilon), \frac{1}{w}\right) = (1 + o(1))\frac{\sqrt[2]{|a_n|}}{\pi\rho}r^\rho, \qquad \rho = \frac{n+2}{2} = \rho(w).$$

Let $p(w)$ denote the number of the set of all $j \in \{0, 1, \ldots, n+1\}$ with the property that for some $\varepsilon > 0$, $w(z)$ has only finitely many zeros in $V_j(\varepsilon)$. Clearly $0 \leq p(w) \leq n+2$. Then for any $\varepsilon > 0$, all but finitely many zeros of $w(z)$ lie on $n + 2 - p(w)$ of all $V_j(\varepsilon)$'s $(0 \leq j \leq n+1)$ and

$$\delta(0, w) = \Delta(0, w) = \frac{p(w)}{4\rho - p(w)}, \tag{3}$$

where $\Delta(0, w)$ is the deficiency in the sense of Valiron.

It is obvious that $\delta(0, w) = \Delta(0, w) = 0$ if and only if $p(w) = 0$; i.e., all but a finite number of zeros of $w(z)$ are equally attracted to the system of rays $\{\theta_0, \ldots, \theta_{n+1}\}$ with $\theta_{j+1} - \theta_j = \frac{\pi}{\rho} = \frac{2\pi}{n+2}$ $(j = 0, 1, \ldots, n+1)$, and that $\delta(0, w) = \Delta(0, w) = 1$ if and only if $p(w) = n+2$, i.e., 0 is a Picard exceptional value (PEV) of $w(z)$. Corollary 2 in [10] tells us that SDE (1) has at most $n+2$ solutions with positive Nevanlinna deficiency at zero without considering constant multiple. However, if w_1 and w_2 are a fundamental system (FS) of SDE (1) and $A(z)$ is not a constant, then

$$p(w_1) + p(w_2) \leq 2n + 2.$$

This assertion comes from the following result.

Theorem 1. *Let w_1 and w_2 be two independent solutions of SDE* (1). *If $A(z)$ is not a constant, then the zeros of $E = w_1 w_2$ are attracted to a system of rays $D_0 = D(\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{im})$ $(0 \leq i_1 < i_2 < \cdots < i_m \leq n+1)$ having the same form as in* (2) *with the properties that for each θ_{i_k} and arbitrary small $\varepsilon > 0$, $V_{i_k}(\varepsilon)$ contains infinitely many zeros of E, $\varpi(D_0) = \frac{n+2}{2} = \rho(w_1) = \rho(w_2)$, and $m \geq 2$.*

It is easy from the above theorem to see that if SDE (1) has a FS, the zeros of which are attracted to a straight line which goes through the origin, then $A(z)$ is a constant. This result is essentially due to Gundersen [10].

What about the distribution of zeros of solutions of general linear differential equation (GDE)

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f = 0, \tag{4}$$

with all coefficients a_j's being polynomials? Many excellent results corresponding to those concerning SDE (1) have been listed; please refer to Frank [7], Steinmetz [13], and Brüggemann [4]. The following is well known and comes from the theory of asymptotic integration (see, e.g., Brüggemann [5]):

GDE (4) has n linearly independent formal solutions

$$w_j^0 = \exp P_j(z) z^{\rho_j} [\log z^{1/p}]^{m_j} Q_j(z, \log z), \qquad 1 \leq j \leq n,$$

where $P_j(z)$ is a polynomial in $z^{1/p}$, $m_j \in N_0$, $\rho_j \in C$, and Q_j is a polynomial in $\log z$ over the field of formal series $\sum_{s \in N_0} a_s z^{-s/p}$, $Q_j(z, \log z) = 1 + O(1/\log z^{1/p})$, as $r \to \infty$.

Then given a ray $\arg z = \theta$, there exists a sufficiently small $h > 0$ (which depends on θ) such that $\{w_j^0 | 1 \leq j \leq n\}$ represents a fundamental system of GDE (4) in the sector $S : |\arg z - \theta| < h$.

Let $w(z)$ be a nontrivial solution of GDE (4). Then

$$w = c_1 w_{i_1}^0 + c_2 w_{i_2}^0 + \cdots + c_m w_{i_m}^0, \qquad c_j \neq 0,\ 1 \leq i_1 < \cdots < i_m \leq n.$$

Definition. A ray $\arg z = \theta \in \mathfrak{R}$ is called a Stokes ray of a nontrivial solution w of GDE (4) if for some δ, $0 < \delta < h$, there exist P_{i_v} and P_{i_k} with $P_{i_v} \neq P_{i_k}$ such that for $\theta < \varphi < \theta + \delta$ and every $P_{i_s} \neq P_{i_v}$ we have

$$\text{(5)} \qquad \operatorname{Re}(P_{i_v}(re^{i\varphi}) - P_{i_s}(re^{i\varphi})) \to +\infty$$

as $r \to +\infty$ and P_{i_k} has the same property for $\theta - \delta < \varphi < \theta$. And further if $\rho(w) = \deg(P_{i_v} - P_{i_k})$, then $\arg z = \theta$ is called a Stokes ray of w of order $\rho(w)$.

It follows from the proof of Lemma 1 in [5] that a ray $\arg z = \theta$ is a Stokes ray of w of order ρ if and only if for some $c > 0$,

$$n(r, S, w) = cr^{\rho(w)}(1 + o(1)),$$

where S is an arbitrary small sector containing the ray $\arg z = \theta$. Hence from Steinmetz [13, Corollary 1], we can obtain the following

Theorem 2. *Let $w(z)$ be a transcendental solution of GDE (4) having Stokes rays $\arg z = \theta_j$, $1 \leq j \leq m$, of order $\rho(w)$. Then the number of zeros of $w(z)$ in $|z| \leq r$, but outside the logarithmic strips $|\arg z - \theta| < A(\log^+ |z|)/|z|^{1/p}$ for $\theta = \theta_1, \ldots, \theta_m$ (A a sufficiently large constant) is $O(r^{\rho - \varepsilon})$ for some $\varepsilon > 0$.*

Hence the zeros of any transcendental solution of GDE (4) are attracted to a system of its Stokes rays of order ρ. Define the indicator function $h_w(\theta)$ of w by

$$\text{(6)} \qquad h_w(\theta) := \limsup_{r \to \infty} \frac{\log |w(re^{i\theta})|}{r^{\rho(w)}} \qquad (\theta \in \mathfrak{R}).$$

Let w be a solution of order $\rho(w)$, $0 < \rho(w) < \infty$, of GDE (4). Theorem 2 in [13] asserts that

$$\text{(7)} \qquad \delta(0, w) = \Delta(0, w) = 1 - \frac{\int_0^{2\pi} h_w(\varphi) d\varphi}{\int_0^{2\pi} h_w^+(\varphi) d\varphi},$$

where $h_w^+(\varphi) = h_w(\varphi)$, when $h_w(\varphi) \geq 0$, and otherwise $h_w^+(\varphi) = 0$.

Dietrich [6] (cf. Brüggemann [5, Theorem C]) pointed out that

$$h_w(\theta) = 0 \quad \text{or} \quad |a| \cos(\arg a + \rho\theta), \qquad \varphi \leq \theta < 2\pi + \varphi,$$

where $a \in C \setminus \{0\}$ and $\rho = \rho(w)$. Hence if $h_w(\theta) = |a| \cos(\arg a + \rho\theta)$, $\varphi \leq \theta < 2\pi + \varphi$, and ρ is an integer, it follows from (7) that $\delta(0, w) =$

$\Delta(0, w) = 1$, i.e., 0 is a Borel exceptional value (BEV) of w (see [13, Corollary 2]). But it is clear that if ρ is not an integer, the above result does not always hold true. In fact, $Ai(z)$, which is a solution of the well-known Airy differential equation $w'' - zw = 0$, possesses the asymptotic form

$$Ai(z) \sim \frac{1}{2\pi^{1/2} z^{1/4}} \exp\left[-\frac{2z^{3/2}}{3}\right] \sum_{k=0}^{\infty} (-1)^k c_k \left(\frac{2z^{3/2}}{3}\right)^{-k} \tag{8}$$

in $|\arg z| < \pi$ (cf. [12, p. 188]). It is obvious that

$$h_{Ai(z)}(\theta) = -\frac{2}{3}\cos\frac{3}{2}\theta, \qquad -\pi < \theta < \pi, \tag{9}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h_{Ai(z)}(\theta) d\theta = \frac{4}{9\pi}, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} h^{+}_{Ai(z)}(\theta) d\theta = \frac{8}{9\pi},$$

so that $\delta(0, Ai(z)) = \Delta(0, Ai(z)) = \frac{1}{2}$. This result can also be found in [10, p. 288] and [13, Theorem 2].

Our results will be

Theorem 3. *Let $w(z)$ be a transcendental solution of GDE (4) having m Stokes rays $\arg z = \theta_j$, $1 \le j \le m$, of order ρ. If $\delta(0, w) = \Delta(0, w) = 0$, then $\rho(w) \le \Theta(D)$ and further $2\rho(w) \le m$.*

Combining Theorem 3 with a well-known result [14] (cf. [8, Lemma 2]), we have that if all the solutions of GDE (4) have order at most $\rho(w)$, then

$$\deg a_j \le (n-j)\left[\min\left\{\Theta(D), \frac{m}{2}\right\} - 1\right], \qquad 1 \le j \le m.$$

And the following are two easy consequences of the above result.

Corollary 1. *If GDE (4) has a transcendental solution $w(z)$ with the property that the zeros of $w(z)$ are attracted to $D = D(\theta_1, \theta_2)$ and $\Delta(0, w) = \delta(0, w) = 0$ and all the solutions of GDE (4) have no order greater than $\rho(w)$, then all the coefficients a_j $(1 \le j \le n-1)$ are identically constant.*

Corollary 2. *Let GDE (4) have a FS $\{w_1, \dots, w_n\}$ with the property that for each j, $1 \le j \le n$, the zeros of $w_j(z)$ are attracted to $D = D(\theta_{1j}, \theta_{2j})$ and $\Delta(0, w_j) = \delta(0, w_j) = 0$; then all the coefficients a_j $(1 \le j \le n-1)$ are identically constant.*

In §2, we shall give the proof of our reults. In §3, we shall discuss the incompleteness in the proof of a conjecture of Hellerstein and Rossi given by Brüggemann in [5].

We conclude this section with

Remarks. (i) It is easy to see that in the case of SDE (1), "$\delta(0, w) = \Delta(0, w) = 0$" if and only if $\rho(w) = \Theta(D)$, $m = 2\rho$, so the condition in Theorem 3 is sharp and the estimates there are the best possible. We easily see from the discussion of the Airy DE that Corollary 1 is sharp.

(ii) Theorem 3 remains true, even if GDE has rational coefficients.

(iii) Theorem 1 is also sharp. Actually, let ω be a nonreal cube root of unity. Then $Ai(z)$ and $Ai(\omega z)$ are two independent solutions of Airy DE, and all the zeros of $E = Ai(z)Ai(\omega z)$ lie only on two rays the minimum argument of which equals to $2\pi/3$.

2. Proof of theorems

In the proof of Theorem 1, we need the following result, which is Theorem 3.6 in [17, p. 204].

Lemma. *Let $f(z)$ be a function meromorphic in the complex plane with finite lower order and a nonzero and finite deficient value. Let $D = D(\phi_1, \phi_2, \dots, \phi_n)$ be a system of rays $D = \bigcup_{j=1}^{n}\{z;\ \arg z = \phi_j\}$, $0 \le \phi_1 < \cdots < \phi_n < \phi_{n+1} = \phi_1 + 2\pi$. If for arbitrary $\varepsilon > 0$, we have*

$$\limsup_{r\to\infty} \frac{\log^+ n(r, G(D, \varepsilon), f = X)}{\log r} = 0 \qquad (X = 0, \infty),$$

then $\rho(f) \le \omega(D)$.

Now we go back to the proof of Theorem 1. Let $\rho = \rho(w_1) = \rho(w_2) = \frac{n+2}{2}$ and $h = w_1/w_2$. It follows from Lemma 2 in [10] that there exists a system of m rays

$$D_0 = D(\theta_{i_1},\ \theta_{i_2}, \dots, \theta_{i_m}) \qquad (0 \le i_1 < \cdots < i_m \le n+1)$$

defined as in (2) such that for arbitrary $\varepsilon > 0$, $h(z)$ only has finitely many zeros and poles in the region $G(D_0, \varepsilon)$ corresponding to the system D_0, but it has an infinite number of zeros or poles in each $V_{i_k}(\varepsilon)$ corresponding to ray $\arg z = \theta_{i_k}$. Obviously, $\rho \ge \omega(D_0)$. Lemma 3 in [10] shows that there exist at most $n+2$ distinct values $b_1, \dots, b_{n+2}$ such that $\sum_{k=1}^{n+2} \delta(b_k, h) = 2$. We shall treat two cases below.

(I) There is a finite b_k $(\ne 0)$ with $\delta(b_k, h) > 0$. Then applying Lemma to $h(z)$, we have $\rho = \rho(h) \le \omega(D_0)$. Hence $\rho = \omega(D_0)$, and further if $m = 1$, then $\rho = \omega(D_0) = \frac{1}{2}$, which contradicts $\rho = (n+2)/2 > \frac{1}{2}$. Consequently, $m \ge 2$.

(II) $\delta(0, h) = \delta(\infty, h) = 1$. Combining Lemma 3 and (4.9), both in [10], we have $T(r, h) = O(T(r, w_i))$ $(i = 1, 2)$, and therefore $N(r, 1/w_i) = o(T(r, w_i))$ $(i = 1, 2)$. The application of (4.8) in [10] to w_i $(i = 1, 2)$ implies that $p(w_1) = p(w_2) = n+2$, i.e., 0 is a PEV of w_1 and w_2, which contradicts $A(z)$ being nonconstant.

Thus Theorem 1 follows.

Proof of Theorem 3. Here we only prove that $\theta_2 - \theta_1 \le \pi/\rho(w)$, for by the same reasoning, we can deduce that $\theta_{j+1} - \theta_j \le \pi/\rho(w)$, and therefore $\rho(w) \le \Theta(D)$. Suppose that $\theta_2 - \theta_1 > \pi/\rho(w)$. Choose a sufficiently small number $\delta > 0$ such that $\theta_2 - \theta_1 - 6\delta > \pi/(\rho(w) - \delta)$. Define

$$k := \pi/(\beta - \alpha), \qquad \theta_1 + \delta < \alpha < \theta_1 + 2\delta,\ \theta_2 - 2\delta < \beta < \theta_2 - \delta,$$

and

$$\gamma_1 := \theta_1 + 3\delta, \qquad \gamma_2 := \theta_2 - 3\delta.$$

Applying the angle Nevanlinna formula [9, p. 41] to the angle $S = \{z;\ \alpha < \arg z < \beta\}$, we have

$$
\begin{aligned}
\mathrm{I}+\mathrm{II} &:= \frac{1}{\pi}\int_1^r \left(\frac{r}{t}\right)^k \left(\log^+ \frac{1}{|w(te^{i\alpha})|} + \log^+ \frac{1}{|w(te^{i\beta})|}\right)\frac{dt}{t} \\
&\quad + \left[4m\left(r, \frac{1}{w}\right) + 2\int_1^r n\left(t, S, \frac{1}{w}\right)\left(\left(\frac{r}{t}\right)^k + \left(\frac{t}{r}\right)^k\right)\frac{dt}{t}\right] \\
&\geq \frac{2}{\pi}\int_\alpha^\beta \log^+ |w(re^{i\varphi})| \sin k(\varphi-\alpha)\, d\varphi \\
&\geq \frac{2\sin k\delta}{\pi}\int_{\gamma_1}^{\gamma_2} \log^+ |w(re^{i\varphi})|\, d\varphi .
\end{aligned} \tag{10}
$$

Since by Theorem 2, $n(r, S, \frac{1}{w}) < Kr^{\rho-\varepsilon}$ for some $K > 0$, it is obvious that $\mathrm{II} = o(T(r, w))$,

$$
\begin{aligned}
\int_{\theta_1+\delta}^{\theta_1+2\delta} d\alpha \int_{\theta_2-2\delta}^{\theta_2-\delta} d\beta (\mathrm{I}) &\leq \frac{2}{\pi}\int_1^r m\left(t, \frac{1}{w}\right)\left(\frac{r}{t}\right)^k \frac{dt}{t} \\
&= \frac{2}{\pi}\int_1^r o(r^\rho)\left(\frac{r}{t}\right)^k \frac{dt}{t} = o(r^\rho) = o(T(r, w))
\end{aligned}
$$

as $r \to +\infty$, since $T(r, w) \sim cr^\rho$ for a $c > 0$. Thus, from (10) we have

$$
\int_{\gamma_1}^{\gamma_2} \log^+ |w(re^{i\varphi})| d\varphi = o(T(r, w)) \tag{11}
$$

as $r \to +\infty$.

On the other hand, from (33) in [13], we have

$$
\log |w(re^{i\varphi})| = h_w(\varphi) r^\rho + O(r^{\rho-\varepsilon}) \tag{12}
$$

uniformly as $r \to \infty$, $\gamma_1 \leq \varphi \leq \gamma_2$, except possibly outside a set $E(r) \subset [0, 2\pi]$ with $\operatorname{mes} E(r) = O(r^{-\varepsilon})$. Since $\arg z = \theta_i$ $(i = 1, 2)$ are two Stokes rays of ρ order of w, $h_w(\varphi) \neq 0$ in a set of φ with a positive measure and, further, for some $a > 0$,

$$
\int_{\gamma_1}^{\gamma_2} \log^+ |w(re^{i\varphi})| d\varphi = ar^\rho + O(r^{\rho-\varepsilon}),
$$

which contradicts (11). Hence $\theta_2 - \theta_1 \leq \pi/\rho(w)$ and thus $\rho(w) \leq \Theta(D)$. Namely, for each j, we have

$$
\frac{\pi}{\theta_{j+1}-\theta_j} \geq \rho(w) \quad \text{and} \quad \frac{\pi}{\rho(w)} \geq \theta_{j+1} - \theta_j ,
$$

so

$$
\frac{m\pi}{\rho(w)} \geq \sum_{j=1}^m (\theta_{j+1} - \theta_j) = 2\pi.
$$

Therefore, $m \geq 2\rho(w)$. Thus Theorem 3 follows. □

3. Brüggemann's proof of the Hellerstein-Rossi conjecture

Frank [7] considered the number of zeros of a fundamental system of GDE (4) and proved that GDE (4) possesses no fundamental system, each element of which has only finitely many zeros, i.e., has PEV 0, unless GDE (4) can be transformed to another GDE with constant coefficients. Following Frank's work, the discussions of this subject have developed in two directions.

One is the Frank-Wittich conjecture (cf. [7, 2, 16]) that the result of Frank remains true if PEV 0 is replaced by BEV 0. This has been proved independently by Brüggemann [4] and Steinmetz [14]; the other is the Hellerstein-Rossi conjecture (cf. [3, Problem 2.72]) that the conclusion of Frank still holds if each solution of a FS of GDE (4) has only finitely many nonreal zeros instead of "PEV 0".

Brüggemann [5] offered a proof of the Hellerstein-Rossi conjecture, but his proof seems to be incomplete. The conclusion he obtained in [5] can actually be stated as follows:

Assume that there is a FS $\{w_1, \dots, w_n\}$ of GDE (4) such that for each j, $1 \le j \le n$,

$$\limsup_{r\to\infty} \frac{\log^+ n_{\mathrm{NR}}(r, w_j)}{\log r} < \rho(w_j), \tag{13}$$

where $n_{\mathrm{NR}}(r, w_j)$ denotes the number of nonreal zeros of w_j in $|z| \le r$ and at least one w_k with $\rho(w_k) = \max_{1\le j\le n} \rho(w_j)$ and without BEV 0 exists. Then all the a_j $(0 \le j \le n-1)$ are constant.

However, Lemmas 2 and 3 in [5], which play an important role in the proof of the above assertions, are incomplete. Actually, Lemma 2 there asserts that any transcendental solution of GDE (4) which has only finitely many nonreal zeros must be of integer order. That certainly is not true, for $Ai(-z)$ is a solution of $w'' + zw = 0$ with only positive real zeros, but it is well known that $\rho(Ai(-z)) = \frac{3}{2}$.

Let us analyse the proof of Lemma 2 in [5]. The case when the solution w_l has only one essential asymptotic change in the sense of Brüggemann [5], which indeed may occur, was not considered there. Hence Lemma 3 in [5] is also incomplete, for the w_l may have only positive or negative real zeros, but, for this case, the form of the indicator function $h_{w_l}(\theta)$ was not listed.

Transcendental entire functions with integer order having only positive real zeros, of course, do exist. A natural problem is raised:

Does there exist a GDE (4) with nonconstant coefficients possessing a transcendental solution with the property that all but a finite number of its zeros are positive real numbers and its order is an integer?

This is closely related to the proof of the Hellerstein-Rossi conjecture by Brüggemann [5]. In the case of SDE (1) when $A(z)$ is a real polynomial, the answer to the above problem is negative (cf. [10, Corollary 3]).

In summary, Brüggemann [5] only proved the following:

Theorem A. *Assume that there is a FS $\{w_1, \dots, w_n\}$ of GDE (4) such that each w_j has either BEV 0 or exactly two essential asymptotic changes at $\arg z = 0$, π and at least one w_k exists among w_j having not BEV 0 such that $\rho(w_k) = \max_{1\le j\le n} \rho(w_j)$. Then all the a_j $(1 \le j \le n-1)$ are constant.*

ACKNOWLEDGMENTS

The author thanks Professor G. Frank, from whom I learned of Brüggemann's works, Professor D. Drasin for his helpful discussions and suggestions in Tian

Jin, China, and Professor I. Laine for his warm encouragement in communication. He is also grateful to Professor Baernstein for his help in improving the exposition of this paper. Finally, the author thanks the referee for information about a recent paper by Steimetz (Proc. Amer. Math. Soc. **117** (1993), 355 – 358) in which he proves, by different methods, a result which contains Theorem 1.

REFERENCES

1. S. B. Bank and I. Laine, *On the oscillation theory of* $f'' + Af = 0$ *where* A *is entire*, Trans. Amer. Math. Soc. **273** (1982), 351–363.
2. S. B. Bank, G. Frank, and I. Laine, *Über die Nullstellen von Lösungen linearer Differentialgleichungen*, Math. Z. **183** (1983), 355–364.
3. D. A. Brannan and W. K. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **21** (1989), 1–35.
4. F. Brüggemann, *On the zeros of fundamental systems of linear differential equations with polynomial coefficients*, Complex Variables **15** (1990), 159–166.
5. ——, *On solutions of linear differential equations with real zeros; Proof of a conjecture of Hellerstein and Rossi*, Proc. Amer. Math. Soc. **113** (1991), 371–379.
6. V. Dietrich, *Über die annahme der möglichen wachstumsordnungen und typen bei linearen differentialgleichungen*, Habilitationsschrift, RWTH, Aachen, 1989.
7. G. Frank, *Picardsche ausnahmewerte bei Lösungen linear differentialgleichungen*, Manuscripta Math. **2** (1970), 181–190.
8. S. Gao and J. K. Langley, *On the zeros of certain linear differential polynomials*, J. Math. Anal. Appl. **53** (1990), 159–178.
9. A. A. Gol′dberg and I. V. Ostrovskii, *Theory of distribution of the values of meromorphic functions*, Nauka, Moscow, 1970. (Russian)
10. G. G. Gundersen, *On the real zeros of solutions of* $f'' + A(z)f = 0$ *where* $A(z)$ *is entire*, Ann. Acad. Sci. Fenn. Ser. A.I. Math. **11** (1986), 275–294.
11. E. Hille, *Ordinary differential equations in the complex domain*, Wiley, New York, 1972.
12. R. B. Paris and A. D. Wood, *Asymptotics of high order differential equations*, Wiley, New York, 1986.
13. N. Steinmetz, *Exceptional values of solutions of linear differential equations*, Math. Z. **201** (1989), 317–326.
14. ——, *Linear differential equations with exceptional fundamental sets*, Analysis **11** (1991), 119–127.
15. H. Wittich, *Zur kennzeichnung linearer differentialgleichungen mit konstanten koeffizienten*, Festband zum 70, Geburtstag von Rolf Nevanlinna (H. P. Künzi and A. Pfluger, eds.), Springer-Verlag, Berlin, 1966.
16. ——, *Bemerkung zur Zerlegung linearer differentialoperataren mit ganzen koeffizienten*, Math. Nachr. **39** (1969), 363–372.
17. G. H. Zhang, *Theory of meromorphic and entire functions*, Science Press, Beijing, 1986. (Chinese)

DEPARTMENT OF MATHEMATICS, TSING HUA UNIVERSITY, 100084, BEIJING, PEOPLE'S REPUBLIC OF CHINA