MENGER MANIFOLDS HOMEOMORPHIC TO THEIR $n$-HOMOTOPIE KERNELS

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Abstract. We give a necessary and sufficient condition that an $(n + 1)$-dimensional Menger manifold ($\mu^{n+1}$-manifold) is homeomorphic to its $n$-homotopy kernel. Such a $\mu^{n+1}$-manifold is called a $\mu_\infty^{n+1}$-manifold. We also prove the following results:

1. Each homeomorphism between two $Z$-sets in a $\mu_\infty^{n+1}$-manifold $M$ extends to an ambient homeomorphism of $M$ onto itself if it is $n$-homotopic to $\text{id}$ in $M$.

2. An $n$-homotopy equivalence between two $\mu_\infty^{n+1}$-manifolds is $n$-homotopic to a homeomorphism.

3. Each map from a $\mu_\infty^{n+1}$-manifold into a $\mu^{n+1}$-manifold is $n$-homotopic to an open embedding.

Introduction

All spaces considered in this paper are assumed to be locally compact separable metrizable and maps are continuous. In [Be], Bestvina introduced Menger manifolds and established the characterization theorem of such manifolds. An $(n + 1)$-dimensional Menger manifold is a topological manifold modeled on the $(n + 1)$-dimensional universal Menger compactum $\mu^{n+1}$, which is also called a $\mu^{n+1}$-manifold. In [Ch3], Chigogidze introduced the notion of the $n$-homotopy kernel of a $\mu^{n+1}$-manifold and proved the following classification theorem for $\mu^{n+1}$-manifolds: Two $\mu^{n+1}$-manifolds have the same $n$-homotopy type if and only if their $n$-homotopy kernels are homeomorphic. There are close relations between Hilbert cube manifold ($Q$-manifold) theory, and Menger manifold theory, and the $n$-homotopy kernel of a $\mu^{n+1}$-manifold plays the role of the product $X \times [0, 1)$ of a $Q$-manifold $X$ with $[0, 1)$. It is said that $X$ is $[0, 1)$-stable if it is homeomorphic to $(\cong) X \times [0, 1)$.

Wong [Wo] showed that a $Q$-manifold $X$ is $[0, 1)$-stable if and only if $X$ is properly contractible to $\infty$; that is, for any compactum $K$ in $X$ there is a proper map $j_K : X \to X \setminus K$ which is properly homotopic to $\text{id}_X$. Replacing a proper homotopy with a proper $n$-homotopy, we have the notion of properly $n$-contractible to $\infty$. Moreover we say that $X$ is properly locally $(n)$-contractible...
at \( \infty \) if for any compactum \( K \subset X \) there is a compactum \( L \subset X \) with \( K \subset L \) such that for each compactum \( L' \subset X \) with \( L \subset L' \) there exists a proper map \( j_{L'} : X \setminus L \to X \setminus L' \) which is properly \((n\text{-})\)homotopic to \( id_{X \setminus L} \) in \( X \setminus K \).

In this paper we define \( \mu_{\infty}^{n+1} \)\text{-}manifolds as \( \mu^{n+1} \)\text{-}manifolds which are properly \( n\text{-}contractible to \( \infty \) and properly locally \( n\text{-}contractible at \( \infty \) and show the following characterization theorem for \( \mu_{\infty}^{n+1} \)\text{-}manifolds.

**Theorem I.** Let \( M \) be a \( \mu_{\infty}^{n+1} \)\text{-}manifold. Then \( M \) is a \( \mu_{\infty}^{n+1} \)\text{-}manifold if and only if \( M \) is homeomorphic to its \( n\text{-}homotopy kernel \( \text{Ker}(M) \).

We will show that two \( n\text{-}homotopic proper maps into a \( \mu_{\infty}^{n+1} \)\text{-}manifold are properly \( n\text{-}homotopic (see Lemma 3.1). Thus we can remove the requirement of an \( n\text{-}homotopy between \( \mu_{\infty}^{n+1} \)\text{-}manifold to be proper, whence we obtain the following \( Z\text{-}set unknotting theorem for \( \mu_{\infty}^{n+1} \)\text{-}manifolds.

**Theorem II.** Each homeomorphism between two \( Z\text{-}sets in a \( \mu_{\infty}^{n+1} \)\text{-}manifold \( M \) extends to an ambient homeomorphism of \( M \) onto itself if it is \( n\text{-}homotopic to \( \text{id} \) in \( M \).

From Theorem 2.2 in [Ch3], it follows that two \( \mu_{\infty}^{n+1} \)\text{-}manifolds of the same \( n\text{-}homotopy type are homeomorphic. Similarly to [C1, Theorem 5], we can clarify the relation between \( n\text{-}homotopy equivalences and homeomorphisms, that is:

**Theorem III.** An \( n\text{-}homotopy equivalence between two \( \mu_{\infty}^{n+1} \)\text{-}manifolds is \( n\text{-}homotopic to a homeomorphism.

Moreover, similarly to \([0,1)\text{-}stable Q\text{-}manifolds [C1, Lemma 3.6], we can strengthen the open embedding theorem [Ch 2, Ch 3].

**Theorem IV.** Each map from a \( \mu_{\infty}^{n+1} \)\text{-}manifold into a \( \mu^{n+1} \)\text{-}manifold is \( n\text{-}homotopic to an open embedding.

1. Preliminaries

We say two (proper) maps \( f, g : X \to Y \) are (properly) \( n\text{-}homotopic (notation: \( f \simeq_n g \), \( f \simeq_p g \), respectively) if, for any (proper) map \( \alpha : Z \to X \) from a space \( Z \) with \( \dim Z \leq n \) into \( X \), the compositions \( f\alpha \) and \( g\alpha \) are (properly) homotopic in the usual sense. The notion of \( n\text{-}homotopy equivalence is defined in the obvious way.

**Proposition 1.1 [Hu].** Let \( f : X \to Y \) be a map, where \( \dim X \leq n \) and \( Y \) is LC\(^n\). Then for any open cover \( \mathcal{U} \) of \( Y \), there are maps \( \phi : X \to P \) and \( \psi : P \to Y \) such that \( f \) and \( \psi\phi \) are \( \mathcal{U}\text{-}homotopic, where \( P \) is a locally finite polyhedron with \( \dim P \leq n \). In particular, we can choose \( \psi \) as a proper map.

Let us recall that a map \( f : X \to Y \) is said to be \( n\text{-}invertible \) if for any space \( Z \) with \( \dim Z \leq n \) and any map \( \alpha : Z \to Y \) there exists a map \( \beta : Z \to X \) such that \( f\beta = \alpha \).

**Proposition 1.2 [Ch2].** Every \( \mu^{n+1} \)\text{-}manifold admits a proper \((n + 1)\text{-}invertible UV\(^n\)\text{-}surjection onto a Q\text{-}manifold.

**Proposition 1.3 [Ch3].** Two \( \mu^{n+1} \)\text{-}manifolds admitting proper UV\(^n\)\text{-}surjections onto the same LC\(^n\)\text{-}space are homeomorphic.
The following theorem is due to Bestvina [Be], where it is stated in terms of \( \mu \)-homotopy. However, as is known [Ch1], the notion of \( \mu \)-homotopy coincides with one of \( n \)-homotopy for maps between locally compact \( LC^n \)-spaces of dimension at most \( n + 1 \).

**Theorem 1.1 (Z-set unknotting theorem).** Let \( M \) be a \( \mu^{n+1} \)-manifold and \( f : A \to B \) be a homeomorphism between Z-sets in \( M \). If \( f \simeq \mu \text{id}_A \) in \( M \), then \( f \) extends to a homeomorphism \( h : M \to M \).

An \( n \)-homotopy kernel of a \( \mu^{n+1} \)-manifold \( M \) is defined to be the complement \( M \setminus f(M) \) of the image of an arbitrary Z-embedding \( f : M \to M \) with \( f \simeq \mu \text{id}_M \). Using the Z-set unknotting theorem, two \( n \)-homotopy kernels are homeomorphic by an ambient homeomorphism of \( M \) onto itself. By \( \text{Ker}(M) \), we denote a representative of \( n \)-homotopy kernels of \( M \). The following proposition is actually proved in [Ch3].

**Proposition 1.4.** For each \( \mu^{n+1} \)-manifold \( M \) there exists a proper \( (n + 1) \)-invertible \( UV^n \)-surjection \( f_n : \mu^{n+1} \to Q \) satisfying the following condition:

\[(*) \quad f_n^{-1}(X) \text{ is a } \mu^{n+1} \text{-manifold for any locally compact } LC^n \text{-space } X \subset Q.\]

**Theorem 1.2 [Dr].** There exists an \( (n+1) \)-invertible \( UV^n \)-surjection \( f_n : \mu^{n+1} \to Q \) satisfying the following condition:

\[(*) \quad f_n^{-1}(X) \text{ is a } \mu^{n+1} \text{-manifold for any locally compact } LC^n \text{-space } X \subset Q.\]

**Theorem 1.3 [Ch4].** For each locally finite polyhedron \( K \), there exists a proper \( (n + 1) \)-invertible \( UV^n \)-surjection \( f_K : M_K \to K \) from a \( \mu^{n+1} \)-manifold \( M_K \) onto \( K \) satisfying the following conditions:

(a) \( f_K^{-1}(L) \text{ is a } \mu^{n+1} \text{-manifold for any closed subpolyhedron } L \text{ of } K; \)
(b) \( f_K^{-1}(Z) \text{ is a Z-set in } f_K^{-1}(L) \text{ for any Z-set } Z \text{ in a closed subpolyhedron } L \text{ of } K.\)

Let \( f : X \to Y \) be a proper map. We say that \( f \) induces an epimorphism of \( j \)-th homotopy groups of ends if for every compactum \( C \subset Y \) there exists a compactum \( K \subset Y \) such that for each point \( x \in X \setminus f^{-1}(K) \) and every map \( \alpha : (S^j, *) \to (Y \setminus K, f(x)) \) there exist a map \( \hat{\alpha} : (S^j, *) \to (X \setminus f^{-1}(C), x) \) and a homotopy \( f\hat{\alpha} \simeq \alpha \text{ rel. } \ast \text{ in } Y \setminus C. \) Also we say that \( f \) induces a monomorphism of \( j \)-th homotopy groups of ends if for every compactum \( C \subset Y \) there exists a compactum \( K \subset Y \) such that for every map \( \hat{\alpha} : S^j \to X \setminus f^{-1}(K) \) with \( f\hat{\alpha} \simeq \ast \text{ in } Y \setminus K \) it follows that \( \hat{\alpha} \simeq \ast \text{ in } X \setminus f^{-1}(C). \) It is said that \( f \) induces an isomorphism of \( j \)-th homotopy groups of ends if \( f \) induces both the epimorphism and monomorphism of \( j \)-th homotopy groups of ends.

**Theorem 1.4 [Be].** Let \( f : M \to N \) be a proper map between \( \mu^{n+1} \)-manifolds. If \( f \) induces an isomorphism of homotopy groups of \( \text{dim } \leq n \) and an isomorphism of homotopy groups of ends of \( \text{dim } \leq n \), then \( f \) is properly \( n \)-homotopic to a homeomorphism.

2. Characterization of \( \mu_{\infty}^{n+1} \)-manifolds

A space \( X \) is said to be properly \( (n-) \)-contractible to \( \infty \) if for any compactum \( K \) in \( X \) there exists a proper map \( j_K : X \to X \setminus K \) which is properly
If for any compactum $K \subset X$ there exists a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \to X \setminus L'$ which is properly $(n)$-homotopic to $\text{id}_{X \setminus L}$ in $X \setminus K$, then a space $X$ is said to be properly locally $(n)$-contractible at $\infty$. It is easy to see that for any space $X$, $X \times [0, 1)$ is properly contractible to $\infty$ and properly locally contractible at $\infty$.

**Lemma 2.1.** Let $X$ be properly $n$-contractible to $\infty$ and properly locally $n$-contractible at $\infty$. Then for each compact cover $\{X_i\}_{i \in \omega}$ of $X$ with $X_i \subset \text{int} X_{i+1}$, there exist a subcover $\{X_{ik}\}_{k \in \omega}, 0 = i_0 < i_1 < i_2 < \cdots$ and a collection of proper maps $\{f_k : X \to X \setminus X_{ik}\}_{k \in \omega}$ such that $f_0 = \text{id}_X$ and $f_{k-1} \simeq^n_p f_k$ in $X \setminus X_{ik-2}$ for $k \geq 1$, where $X_{i_{-1}} = \emptyset$.

**Proof.** For technical reasons we assume that $X_0 = \emptyset$. Let $L_{-2} = L_{-1} = L_0 = \emptyset$. We shall inductively choose integers $0 = i_2 = i_{-1} = i_0 < i_1 < i_2 < \cdots$ and construct compacta $L_{k-1} \subset X_{ik} \subset L_k$ and proper maps $j_k : X \setminus L_{k-2} \to X \setminus X_{ik}$, $k \in \omega$, satisfying the following conditions:

1. $j_0 = \text{id}_X$.
2. For each compactum $M \supset L_k$ there is a proper map $j_M : X \setminus L_k \to X \setminus M$ such that $j_M \simeq^n p \text{id}_{X \setminus L_k}$ in $X \setminus X_{ik-2}$.
3. $j_k \simeq^n p \text{id}_{X \setminus L_{k-2}}$ in $X \setminus X_{ik-2}$.

Let $i_1 = 1$. Since $X$ is properly $n$-contractible to $\infty$ and properly locally $n$-contractible at $\infty$, there exist a proper map $j_1 : X \to X \setminus X_i$ with $j_1 \simeq^n p \text{id}$ and a compactum $L_1 \supset X_1$ satisfying (2). Since $X = \bigcup_{i \in \omega} X_i$ and $X_i \subset \text{int} X_{i+1}$, there exists $i_2 > i_1$ such that $X_{i_2} \supset L_1$. As in the above arguments there exist a proper map $j_2 : X \to X \setminus X_{i_2}$ with $j_2 \simeq^n p \text{id}_X$ and a compactum $L_2 \supset X_{i_2}$ satisfying (2).

Assume that, for $k \geq 2$, $i_0 < i_1 < \cdots < i_k$, $L_k$, and $j_k : X \setminus L_{k-2} \to X \setminus X_{ik}$ have been constructed. Choose $i_{k+1} > i_k$ so that $X_{ik+1} \supset L_k$. Since $X_{ik+1} \supset L_{k-1}$, by the property (2) of $L_{k-1}$, there exists a proper map $j_{X_{ik+1}} : X \setminus L_{k-1} \to X \setminus X_{ik+1}$ such that $j_{X_{ik+1}} \simeq^n p \text{id}_{X \setminus L_{k-2}}$ in $X \setminus X_{ik-1}$. Then put $j_{k+1} = j_{X_{ik+1}}$. Since $X$ is properly locally $n$-contractible at $\infty$, there exists a compactum $L_{k+1} \supset X_{ik+1}$, satisfying (2).

Now define $f_k = j_k \cdots j_0 : X \to X \setminus X_k$ for $k \in \omega$ and observe that the collections of compacta $\{X_{ik}\}_{k \in \omega}$ and maps $\{f_k\}_{k \in \omega}$ are as desired. $\square$

As is stated in the introduction, a $\mu_{\infty+1}$-manifold is a $\mu^{n+1}$-manifold which is properly $n$-contractible to $\infty$ and properly locally $n$-contractible at $\infty$. Theorem I is contained in the following.

**Theorem 2.1 (Characterization).** For a $\mu^{n+1}$-manifold $M$ the following conditions are equivalent:

1. $M$ is a $\mu_{\infty+1}$-manifold.
2. $M \cong \text{Ker}(M)$.
3. There is a proper $(n+1)$-invertible $UV^n$-surjection $f : M \to X$ onto some $[0, 1)$-stable $Q$-manifold $X$.
4. There is a proper $(n+1)$-invertible $UV^n$-surjection $g : M \to Y$ onto a space $Y$ which is properly $n$-contractible to $\infty$ and properly locally $n$-contractible at $\infty$. 

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Proof. We shall prove that \((1) \Rightarrow (2)\). First we shall choose a compact cover \(\{M_i\}_{i \in \omega}\) of \(M\) with \(M_i \subset \text{int} M_{i+1}\), \(i \in \omega\), such that the topological frontier \(\text{Fr} M_i\) is a \(Z\)-set in \(M \setminus \text{int} M_i\). To this end, fix a proper \(UV^n\)-surjection \(g : M \to X\) onto a \(Q\)-manifold \(X\). Then choose a compact cover \(\{X_i\}_{i \in \omega}\) of \(X\) consisting of \(Q\)-manifold with \(X_i \subset \text{int} X_{i+1}\) such that \(\text{Fr} X_i\) is a \(Z\)-set in both \(X_i\) and \(X \setminus \text{int} X_i\), \(i \in \omega\) (see [C2, CS]). For each \(i \in \omega\), by the relative triangulation theorem for \(Q\)-manifolds [C3], we may assume that \(X = P \times Q\), \(X_i = P_i \times Q\), and \(X \setminus \text{int} X_i = P_i' \times Q\) for a locally finite polyhedron \(P\) and closed subpolyhedra \(P_i', P_i'' \subset P\). Let \(f_P : M_P \to P\) be a proper \(UV^n\)-surjection from a \(\mu^{n+1}\)-manifold \(M_P\) onto \(P\) satisfying condition (b) in Theorem 1.3. Since the composition \(\pi_P g : M \to P\) is proper \(UV^n\) (where \(\pi_P : P \times Q \to P\) is the canonical projection), there is a homeomorphism \(k : M_P \to M\) by Proposition 1.3. Then by property (b) of \(f_P\), \(f_P^{-1}(P_i' \cap P_i'')\) is a \(Z\)-set in \(f_P^{-1}(P_i')\) and so is the topological frontier of \(f_P^{-1}(P_i')\). Now let \(M_i = k f_P^{-1}(P_i')\), \(i \in \omega\). Then the compact cover \(\{M_i\}_{i \in \omega}\) of \(M\) is the required one.

By Lemma 2.1, there is a collection of maps \(\{f_i : M \to M \setminus M_i\}_{i \in \omega}\) such that \(f_0 = \text{id}_M\), \(f_i \simeq f_{i+1}\) in \(M \setminus M_{i+1}\) for \(i \in \omega\). Using the \(Z\)-embedding approximation theorem for \(\mu^{n+1}\)-manifolds [Be, 2.3.8], we can choose \(f_i\) as a \(Z\)-embedding for each \(i \in \omega\). Put \(K_i = M \setminus f_i(M)\) for \(i \geq 1\). Then since \(f_i \simeq \text{id}_M\), the definition of \(n\)-homotopy kernels, we have \(K_i \cong \text{Ker}(M)\).

By Theorem 1.1, since \(f_i \simeq f_{i+1}\) is in \(M \setminus M_{i+1}\) and \(\text{Fr} M_{i+1}\) is a \(Z\)-set in \(M \setminus \text{int} M_{i+1}\), there exists a homeomorphism \(h_i : M \to M\) such that \(h_i \circ f_i = f_{i+1}\) and \(h_i|_{M_{i+1}} = \text{id}_M\). Note that \(h_i(K_i) = K_{i+1}\). Now we define \(h : K_1 \to M\) by \(h = \lim_{i \to \infty} h_1 \cdots h_i\). Then \(h|_{K_i}(M_i) = h_{i+2} \cdots h_i|_{M_i}\). In fact, suppose that \(h(x) \neq h_{i+2} \cdots h_i(x)\) for some \(x \in h^{-1}(M_i)\). Then there is an open subset \(U\) of \(\text{int} M_i\) such that \(h(x) \not\in U \subset M_i\) and \(h_{i+2} \cdots h_i(x) \not\in U\). This contradicts the definition of \(h\).

One can easily see that \(h\) is injective. Moreover, since \(M = \bigcup_{i \in \omega} M_i\) and \(h_1 \cdots h_i(K_i) = K_i \supset M_i\), it follows that \(h\) is surjective. To finish the proof, it only remains to note that \(h\) is open. Thus \(h\) is a homeomorphism.

To prove \((2) \Rightarrow (3)\), assume \(M \cong \text{Ker}(M)\). Then, by Proposition 1.4, there is a proper \((n+1)\)-invertible \(UV^n\)-surjection \(g : M \to M \times [0, 1)\). Let \(h : M \to Y\) be a proper \(UV^n\)-surjection onto a \(Q\)-manifold \(Y\) (Proposition 1.2). Since \(Y \times [0, 1)\) is a \([0, 1)\)-stable \(Q\)-manifold, the composition \((h \times \text{id}_{[0, 1)\})g : M \to Y \times [0, 1)\) is the required one.

\((3) \Rightarrow (4)\) is trivial.

Finally we shall show that \((4) \Rightarrow (1)\). Let \(h : M \to X\) be a proper \((n+1)\)-invertible \(UV^n\)-surjection onto a space \(X\) properly \(n\)-contractible to \(\infty\) and properly locally \(n\)-contractible at \(\infty\). Let \(K\) be a compactum in \(M\). Then there exists a compactum \(L'\) in \(X\) with \(h(K) \subset L'\) such that for each compactum \(F'\) with \(L' \subset F'\) there exist proper maps \(h_{(K)} : X \setminus h(K)\) and \(j_{F'} : X \setminus L' \to X \setminus F'\) such that \(h_{(K)} \simeq \text{id}\) in \(X\) and \(j_{F'} \simeq \text{id}\) in \(X \setminus L'\). Let \(L = h^{-1}(L')\) and \(F\) be a compactum containing \(L\). Since \(h\) is proper \((n+1)\)-invertible, there exist proper maps \(i_K : M \to M \setminus K\) and \(j_F : M \setminus L \to M \setminus F\) such that \(hi_K = i'_K h\) and \(hj_F = j'_F h\).
Consider a proper map \( \alpha : Z \to M \setminus L \) (\( \subset M \setminus h^{-1}(h(K)) \)), where \( \dim Z \leq n \). We shall now show that \( j_F \alpha \) is properly homotopic to \( \alpha \) in \( M \setminus K \). From Proposition 1.1, we may assume without loss of generality that \( Z \) is a locally finite polyhedron. Let \( H : (X \setminus L') \times [0, 1] \to X \setminus h(K) \) be a proper homotopy from \( \text{id}_{X \setminus L'} \) to \( j_h(F') \). Then \( H(h \alpha \times \text{id}) : Z \times [0, 1] \to X \setminus h(K) \) is a proper homotopy from \( h \alpha \) to \( j_h(F')h \alpha = j_F h \alpha \). Since \( h |_{M \setminus h^{-1}(h(K))} : M \setminus h^{-1}(h(K)) \to X \setminus h(K) \) is proper \( UV^n \), by [La, §3, Lemma A], there exists a proper homotopy \( F : Z \times [0, 1] \to M \setminus h^{-1}(h(K)) \) from \( \alpha \) to \( j_F \alpha \). Thus \( j_F \simeq_{id} \text{id}_{M \setminus L} \) in \( M \setminus K \). Similarly, we can conclude \( i_F \simeq_{id} \text{id}_M \). \( \square \)

3. PROOFS OF THEOREMS II, III, AND IV

Lemma 3.1. Let \( f : X \to Y \) be a map from a locally compact space \( X \) into a \( LC^n \)-space \( Y \) admitting a proper \((n + 1)\)-invertible \( UV^n \)-surjection onto a space \( Y \times [0, 1] \). Then \( f \) is \( n \)-homotopic to a proper map whenever \( \dim X \leq n + 1 \). Moreover, if \( f \) is a proper map \( n \)-homotopic to a proper map \( g : X \to Y \), then \( f \simeq_n g \).

Proof. Fix a proper map \( p : X \to [0, 1] \), and let \( h : Y \to Y \times [0, 1] \) be a proper \((n + 1)\)-invertible \( UV^n \)-surjection. Let \( q : X \to Y \times [0, 1] \) be the map defined by \( q(x) = (h_{1}(f(x), p(x))) \), where \( h(x) = (h_{1}(x), h_{2}(x)) \), \( x \in X \). Then \( q \) is proper and homotopic to \( h f \). By the \((n + 1)\)-invertibility of \( h \), there is a map \( f' : X \to Y \) such that \( h f' = q \). Note that \( f' \) is proper and \( h f' \simeq h f \). Thus by the lifting property of \( h \) [La, §3, Lemma A], we conclude that \( f \simeq_n f' \).

Next suppose that \( f \) is a proper map \( n \)-homotopic to a proper map \( g : X \to Y \). Let \( \alpha : Z \to X \) be a proper map, where \( \dim Z \leq n \). We shall show that \( f \alpha \simeq_p g \alpha \). By Proposition 1.1, we may assume without loss of generality that \( Z \) is a locally finite polyhedron. Let \( \{Y_i\}_{i \in \omega} \) be a compact cover of \( Y \) with \( Y_0 = \emptyset \) and \( Y_i \subset \text{int} Y_{i+1}, i \in \omega \). Then for each \( i \geq 1 \), let \( Z_i \) be a compact subpolyhedron of \( Z \) such that \( (h f \alpha)^{-1}(W_i) \cup (h g \alpha)^{-1}(W_i) \subset Z_i \subset \text{int} Z_{i+1} \), where \( Z_0 = \emptyset \) and \( Z_1 = Y_1 \times [0, 1 - 2^{-i}] \). Since \( f \simeq_n g \), we can fix a homotopy \( G_0 : Z \times [0, 1] \to Y \) from \( f \alpha \) to \( g \alpha \). For \( k \geq 1 \), we shall inductively construct a homotopy \( G_k : (Z \setminus \text{int} Z_{k+1}) \times [0, 1] \to Y \setminus h^{-1}(W_{2k+1}) \) from the restriction \( f \alpha |_A \) to the one \( g \alpha |_B \) satisfying the following conditions:

\( (1)_k \) \hspace{1cm} \( G_k((Z \setminus \text{int} Z_{2k}) \times [0, 1]) \subset Y \setminus h^{-1}(W_{2k-2}) \);

\( (2)_k \) \hspace{1cm} \( G_k = G_{k-1} \) on \( \text{Fr} Z_{2k-2} \times [0, 1] \).

Let \( F_i : [0, 1] \to [1 - 2^{-i}, 1] \) be the map defined by \( F_i(t) = 1 + (t - 1)2^{-i} \) for each \( i \geq 1 \). Suppose that a homotopy \( G_k : (Z \setminus \text{int} Z_{k+1}) \times [0, 1] \to Y \setminus h^{-1}(W_{2k+1}) \) has been constructed for \( k \in \omega \). Then let \( A_{k+1} = (Z \setminus \text{int} Z_{2k+1}) \setminus \{0, 1\} \cup \text{Fr} Z_{2k+1} \times [0, 1] \) and \( B_{k+1} = (h_g \alpha)^{-1}(W_{2k+1}) \cap (Z \setminus \text{int} Z_{2k+2}) \times [0, 1] \). Since \( A_{k+1} \) and \( B_{k+1} \) are disjoint closed, we can choose \( \beta : (Z \setminus \text{int} Z_{2k+1}) \setminus \{0, 1\} \to [0, 1] \) such that \( \beta(A_{k+1}) = 0 \) and \( \beta(B_{k+1}) = 1 \). Define \( G'_{k+1} : (Z \setminus \text{int} Z_{2k+1}) \times [0, 1] \to Y \setminus h^{-1}(W_{2k+1}) \) by

\( G'_{k+1}(w) = (s_k(w), (1 - \beta(w))t_k(w) + \beta(w)F_{k+1}t_k(w)) \),

where \( h_g \alpha(w) = (s_k(w), t_k(w)), w \in (Z \setminus \text{int} Z_{2k}) \times [0, 1] \). By the lifting property [La], there is a homotopy \( G_{k+1} : (Z \setminus \text{int} Z_{2k+1}) \times [0, 1] \to Y \setminus W_{2k-3} \) from \( f \alpha \) to \( g \alpha \) with \( h_g \alpha = G'_{k+1} \) and \( G_{k+1} = G_k \) on \( A_{k+1} \) (i.e., satisfying \( (2)_{k+1} \)) such that \( G_{k+1} \) satisfies \( (1)_{k+1} \).
We define $H : Z \times [0, 1] \to Y$ by $H = G_k$ on each $(Z_{2k} \setminus \text{int } Z_{2k-2}) \times [0, 1]$. Then $H$ is a well-defined homotopy from $f\alpha$ to $g\alpha$. Note that since $h$ is proper, $\{h^{-1}(W_i)\}_{i \in \omega}$ is a compact cover of $Y$ with $h^{-1}(W_i) \subset \text{int } h^{-1}(W_{i+1})$. Thus it follows from our construction that $H$ is proper. The proof is finished. □

**Proof of Theorem II.** The theorem directly follows from Theorem 1.1 and Lemma 3.1. □

**Lemma 3.2.** If $f : M \to N$ is a proper $n$-homotopy equivalence between $\mu^{n+1}_\infty$-manifolds, then $f$ induces an isomorphism of homotopy groups of ends of dim $\leq n$.

**Proof.** By Theorem 2.1, we can fix proper $(n+1)$-invertible $UV^n$-surjections $g : M \to X \times [0, 1)$ and $h : N \to Y \times [0, 1)$, where $X$ and $Y$ are some $Q$-manifolds. Let $C$ be a compactum in $N$. Then there is a compactum $C'' \subset Y$ such that $C'' \times [0, t'] \supset h(C)$ for some $t' \in (0, 1)$. Since $h$ is proper, $C' = h^{-1}(C'' \times [0, t'])$ is a compactum with $C' \supset C$. Note that, since $f$ is proper, $g(f^{-1}(C'))$ is a compactum in $X \times [0, 1)$. Thus there exists $t_1 \in (0, 1)$ such that $L = \pi_X(g(f^{-1}(C')) \times [0, t_1]) \supset g(f^{-1}(C'))$, where $\pi_X : X \times [0, 1) \to X$ is the canonical projection. Similarly, since $g$ is proper, there exists $t_2 \in (0, 1)$ such that $K' = \pi_Y(h(f(g^{-1}(L))) \times [0, t_2]) \supset h(f(g^{-1}(L)))$, where $\pi_Y : Y \times [0, 1) \to Y$ is the canonical projection. Put $K = h^{-1}(K')$, and let $x_0 \in M \setminus f^{-1}(K)$, $j \leq n$, and $a : (S^j, *) \to (N \setminus K, f(x_0))$. Since $f$ is an $n$-homotopy equivalence, there exists $a_1 : (S^j, *) \to (M, x_0)$ such that $f\alpha_1 \simeq a$ rel. $\ast$. Since $a_1^{-1}(x_0)$ and $a_1^{-1}(L)$ are disjoint closed sets in $S^j$, we can choose a map $\beta : S^j \to [0, 1)$ such that $\beta(a_1^{-1}(x_0)) = 0$ and $\beta(a_1^{-1}(L)) = 1$. Say $g(a_1(x)) = (\pi_X g(a_1(x)), t(x)) \in X \times [0, 1)$, $x \in S^j$. Define $a_2 : (S^j, *) \to (X \times [0, 1), g(x_0))$ by

$$a_2(x) = (\pi_X g a_1(x), (1 - t_1) \cdot t(x) + t_1) \beta(x) + (1 - \beta(x) \cdot t(x)),$$

$x \in S^j$. Clearly $a_2 \simeq a_1$ rel. $\ast$ and $a_2(S^j) \cap L = \emptyset$. Using the lifting property [La] of the proper $UV^n$-surjection $g$, there exists $\tilde{a} : (S^j, *) \to (M, x_0)$ such that $\text{img } \tilde{a} \cap L = \emptyset$ and $\tilde{a} \simeq a_1$ rel. $\ast$. Hence we have $f\tilde{a} \simeq a$ rel. $\ast$ and $f\tilde{a}(S^j) \cap C' = \emptyset$. By the same technique we performed above, we can choose a homotopy so that $f\tilde{a} \simeq a$ rel. $\ast$ in $N \setminus C$.

Next let $\gamma : S^j \to M \setminus f^{-1}(K)$ be a map such that $f\gamma \simeq \ast$ in $N \setminus K$. Since $f$ is an $n$-homotopy equivalence, $g\gamma \simeq \ast$ in $X \times [0, 1)$. By sliding the $(0, 1)$-factor of the homotopy upward as the above, we have $g\gamma \simeq \ast$ in $X \times [0, 1) \setminus L$. By the lifting property of $g$ [La], it follows that $\gamma \simeq \ast$ in $X \setminus f^{-1}(C)$. Thus we conclude that $f$ induces an isomorphism of homotopy groups of ends of dim $\leq n$. □

**Proof of Theorem III.** Let $f : M \to N$ be an $n$-homotopy equivalence between $\mu^{n+1}_\infty$-manifolds. Then by Lemma 3.1 there is a proper map $h : M \to N$ such that $f \simeq h$; consequently, $h$ is a proper $n$-homotopy equivalence. By Lemma 3.2 and Theorem 1.4, $h$ is properly $n$-homotopic to a homeomorphism. Thus $f$ is $n$-homotopic to a homeomorphism. □
Proof of Theorem IV. Let \( f : M \to N \) be a map from a \( \mu_\infty^{n+1} \)-manifold to a \( \mu_\infty^{n+1} \)-manifold. By replacing \( N \) with \( \text{Ker}(N) \), we may assume that \( N \) is also a \( \mu_\infty^{n+1} \)-manifold. By the triangulation theorem for \( \mu_\infty^{n+1} \)-manifold [Dr], we can fix proper \((n + 1)\)-invertible \( UV^n \)-surjections \( g : M \to K \) and \( h : N \to L \), where \( K \) and \( L \) are locally finite polyhedra of dimension at most \( n + 1 \). Then by the \((n + 1)\)-invertibility, \( g \) has a section \( p : K \to M \) (i.e., \( gp = \text{id}_K \)). Since \( N \) is a \( \mu_\infty^{n+1} \)-manifold, by Lemma 3.1, \( f \) is \( n \)-homotopic to a proper map \( f' : M \to N \). Then \( \phi = hf'p : K \to L \) is a proper map. Let \( M(\phi) \) be the mapping cylinder of \( \phi \), that is, a space obtained from the disjoint union \( K \times [0, 1] \oplus L \) by identifying \( (x, 1) \) with \( \phi(x), x \in K \). Define \( c_\phi : M(\phi) \to L \) by \( c_\phi(x, t) = \phi(x), x \in K \). Let \( f_n : \mu^{n+1} \to Q \) be a proper \((n + 1)\)-invertible \( UV^n \)-surjection satisfying the condition \((\ast)\) in Theorem 1.2. Embed \( M(\phi) \) into \( Q \), whence \( f_{n-1}(M(\phi)) \) is a \( \mu^{n+1} \)-manifold. We denote the restriction of \( f_n \) to \( f_{n-1}(M(\phi)) \) by \( f_n \). Observe that \( f_{n-1}(K \times \{0\}) \cong M \) and \( f_{n-1}(L) \cong N \) by Proposition 1.3. We identify \( f_{n-1}(K \times \{0\}), f_{n-1}(L) \) with \( M, N \) respectively. Abusing notation, by \( g : M \to K \times \{0\}, h : N \to L \) we denote the restrictions of \( f_n \) to \( M, N \) respectively. Using the \((n + 1)\)-invertibility of \( h \), we can fix a section \( q : L \to N \) of \( h \). Note that since \( c_\phi f_n : f_{n-1}(M(\phi)) \to L \) and \( h : N \to L \) are proper \( UV^n \)-surjections, \( f_{n-1}(M(\phi)) \cong N \) by Proposition 1.3. Observe that the map \( q c_\phi f_n \) is an \( n \)-homotopy equivalence between \( \mu_\infty^{n+1} \)-manifolds \( f_{n-1}(M(\phi)) \) and \( N \). Then by Theorem III, there is a homeomorphism \( s : f_{n-1}(M(\phi)) \to N \) such that \( s \cong^n q c_\phi f_n \). Note that \( M' = f_{n-1}(K \times \{0, 1\}) \) is open in \( f_{n-1}(M(\phi)) \) and is a \( \mu_\infty^{n+1} \)-manifold by Theorem 2.1. Since the inclusion \( i : M = f_{n-1}(K \times \{0\}) \hookrightarrow M' \) is an \( n \)-homotopy equivalence, by Theorem III, we can choose a homeomorphism \( r : M \to M' \) with \( r \cong^n i \). Then the map \( sr : M \to N \) is an open embedding which is \( n \)-homotopic to \( q c_\phi(f_n)i = q\phi g = qh f'p g \). Since \( pg \cong^n p \text{id}_M \) and \( qh \cong^n p \text{id}_N \), we have \( qhf'p g \cong^n f' \cong^n f \). The proof is finished. \( \square \)

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References


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