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LOCAL MINIMIZERS OF INTEGRAL FUNCTIONALS ARE GLOBAL MINIMIZERS

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ABSTRACT. We show that local minimizers of integral functionals associated with a measurable integrand $f: \Omega \times E \to \mathbb{R} \cup \{\pm\infty\}$ are actually global minimizers. Here $(\Omega, \mathscr{S}, \mu)$ is a measured space with an atomless σ-finite positive measure, E is a separable Banach space, and the integral functional $I_f(x) = \int_\Omega^* f(\omega, x(\omega))\, d\mu$ is defined on $L_p(\Omega, E)$ or, more generally, on some decomposable set of measurable mappings x from Ω into E.

In a minimization problem, it is natural to delineate situations where a local minimizer is a global one. Apart from convex problems, it is usually unexpected that a local minimizer of the given objective function still minimizes it on the whole space where it is defined. Here we show that local minimizers are indeed global ones when the objective function is an integral functional defined on some decomposable space endowed with a suitable topology.

We assume throughout that $(\Omega, \mathscr{S})$ is a measurable space equipped with a positive σ-finite measure μ; we suppose, moreover, that μ is atomless. The underlying spaces X of minimization that we are considering are rather special ones: with E a separable Banach space and its Borel tribe $\mathscr{B}(E)$, we denote by X a set of measurable functions from Ω into E satisfying the following *decomposability property* [2]: for all measurable subsets A of Ω and all x and y in X, the "decomposed" $1_A x + 1_{\Omega\setminus A} y$ still belongs to X (as usual 1_A is the characteristic function of A).

From the topological viewpoint, we suppose that X is endowed with a topology τ for which

$$(1) \qquad (A \in \mathscr{S},\ \mu(A) \to 0) \Rightarrow (1_A y + 1_{\Omega\setminus A} x \underset{\tau}{\longrightarrow} x).$$

The property above is satisfied, for example, if:

$X = L_p(\Omega, E)$ and X is endowed with the Mackey topology $\tau(L_p, L_q)$ $(1/p + 1/q = 1)$;

$X = L_p(\Omega, E)$, with $1 \le p < +\infty$, endowed with the usual norm topology.

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Now let $f\colon \Omega \times E \to \mathbb{R} \cup \{\pm\infty\}$ be a $\mathscr{S} \otimes \mathscr{B}(E)$-measurable integrand [1, 4]; by *integral functional* associated with f, we mean the following functional I_f:

$$x \in X \to I_f(x) := \int_\Omega^* f(\omega, x(\omega))\, d\mu$$

$$\text{[the upper integral of } \omega \to f(\omega, x(\omega))]$$

$$= \inf\left\{\int_\Omega u(\omega)\, d\mu,\ u \in L_1(\Omega, \mathbb{R}),\ u(\omega) \geq f(\omega, x(\omega))\ \mu\text{-a.e.}\right\}.$$

Surprisingly enough, it turns out that under the assumptions displayed above, there is no distinction to be made between local and global minimizers of I_f on X. The main result reads as follows:

Theorem. *Let $\overline{x}$ be a local minimizer of I_f on X, with $I_f(\overline{x}) \in \mathbb{R}$. Then*

$$f(\omega, \overline{x}(\omega)) \leq f(\omega, x(\omega))\ \mu\text{-a.e.} \quad \text{for all } x \in X, \tag{2}$$

so that

$$I_f(\overline{x}) \leq I_f(x) \quad \text{for all } x \in X.$$

Proof. For a given $x \in X$, we intend to show that

$$A := \{\omega \in \Omega\colon f(\omega, \overline{x}(\omega)) > f(\omega, x(\omega))\}$$

is of measure zero. Suppose by contradiction that $\mu(A) > 0$, and for $n \in \mathbb{N}^*$ denote

$$A_n := \left\{\omega \in \Omega\colon f(\omega, \overline{x}(\omega)) \geq f(\omega, x(\omega)) + \frac{1}{n}\right\}. \tag{3}$$

It is clear that $(A_n)_{n\in\mathbb{N}^*}$ is an increasing sequence of measurable subsets such that $\bigcup_{n\in\mathbb{N}^*} A_n = A$. Therefore, we infer from the assumption $\mu(A) > 0$ that there exists $n_0 \in \mathbb{N}^*$ for which $\mu(A_{n_0}) > 0$.

Since the measure μ is atomless, we can find a sequence (B_k) of measurable subsets of A_{n_0} satisfying

$$\mu(B_k) > 0 \text{ for all } k \quad \text{and} \quad \mu(B_k) \xrightarrow[k\to+\infty]{} 0. \tag{4}$$

(This comes from the well-known theorem of Liapunov asserting that the range of an atomless $\mathbb{R}^p$-valued measure is necessarily convex [3].) We set

$$x_k := 1_{B_k} x + 1_{\Omega\setminus B_k} \overline{x}. \tag{5}$$

By our assumptions on the decomposability of X and (1), (4), we have that $x_k \to x$ in X when $k \to +\infty$. As a consequence, we get

$$I_f(x_k) = \int_{B_k}^* f(\omega, x(\omega))\, d\mu + \int_{\Omega\setminus B_k}^* f(\omega, \overline{x}(\omega))\, d\mu,$$

with

$$\int_{B_k}^* f(\omega, x(\omega))\, d\mu \leq \int_{B_k}^* f(\omega, \overline{x}(\omega))\, d\mu - \frac{1}{n_0}\mu(B_k)$$

[because $B_k \subset A_{n_0}$ and (3)]; whence,

$$\begin{aligned} I_f(x_k) &\leq \int_\Omega^* f(\omega, \overline{x}(\omega))\, d\mu - \frac{1}{n_0}\mu(B_k) = I_f(\overline{x}) - \frac{1}{n_0}\mu(B_k) \\ &< I_f(\overline{x}). \end{aligned}$$

But since $x_k \to x$, the above strict inequality enters in contradiction with the fact that $\overline{x}$ is a local minimizer of I_f on X.

Thus $\mu(A) = 0$, i.e. (2), and the inequality $I_f(\overline{x}) \leq I_f(x)$ follows.

REFERENCES

1. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Math., vol. 580, Springer-Verlag, Berlin and New York, 1977.
2. F. Hiai, *Representation of additive functionals on vector valued normed Kothe space*, Kodai Math J. **2** (1979), 300–313.
3. J. Lindenstraus, *A short proof of Liapunov's convexity theorem*, J. Math. Mech. **15** (1966), 971–972.
4. R. T. Rockafellar, *Integral functionals, normal integrands, and measurable selections*, Lecture Notes in Math., vol. 543, Springer-Verlag, Berlin and New York, 1976, pp. 157–206.

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