NUMERICAL MESHES AND COVERING MESHES
OF APPROXIMATE INVERSE SYSTEMS OF COMPACTA

TADASHI WATANABE

Abstract. Mardesic and Rubin (1989) introduced approximate inverse systems of metric compacta by the conditions (A1*)-(A3)*. Mardesic and Watanabe (1988) introduced approximate inverse systems of topological spaces by the conditions (A1)-(A3). In this note we show that any approximate inverse system of metric compacta satisfies (A1)-(A3) if and only if it satisfies (A1*)-(A3)* for some matrices (see Theorem 1).

S. Mardesic and L. Rubin [1] introduced the notion of approximate inverse system \( \mathcal{X} = \{ (X_a, d_a), e_a, p_{aa'}, A \} \) of metric compacta. Hence, \( (A, \leq) \) is a directed preordered infinite set. \( (X_a, d_a) \) is a compactum endowed with a metric \( d_a \), and \( p_{aa'} : X_{a'} \to X_a \) is a mapping defined whenever \( a \leq a' \) and is such that \( p_{aa} \) is the identity mapping. The real numbers \( e_a > 0, a \in A, \) are called numerical meshes. We require the following conditions:

\[ \begin{align*}
(A1)* & \quad (\forall a_2 \geq a_1 \geq a) d_a(p_{aa}, p_{a_1a_2}, p_{aa_2}) \leq e_a. \\
(A2)* & \quad (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a') d_a(p_{aa}, p_{a_1a_2}, p_{aa_2}) \leq \eta. \\
(A3)* & \quad (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall x, x' \in X_{a''}) d_{a''}(x, x') \leq e_{a''} \\
& \quad \text{implies } d_a(p_{a_{a''}}(x), p_{a_{a''}}(x')) \leq \eta. 
\end{align*} \]

In [6] S. Mardesic and T. Watanabe introduced the notion of an approximate inverse system \( \mathcal{X} = \{ X_a, \mathcal{U}_a, p_{aa'} , A \} \) of topological spaces. Here \( A \) is a directed preordered infinite set, \( \mathcal{U}_a \) is a normal open covering of \( X_a, a \in A, \) also called a mesh, and \( p_{aa'} : X_{a'} \to X_a \) is a mapping defined whenever \( a \leq a' \) and is such that \( p_{aa} \) is the identity mapping. We require three conditions (A1)-(A3), which are natural analogues of conditions (A1*)-(A3)*. Before we state these conditions, let us denote that \( \text{Cov}(X) \) is the set of all normal open coverings of a space \( X \). For \( \mathcal{U}, \mathcal{V} \in \text{Cov}(X), \mathcal{V} < \mathcal{U} \) means that \( \mathcal{V} \) refines \( \mathcal{U} \). If \( \mathcal{V} \in \text{Cov}(Y) \) and \( f, f' : X \to Y \) are mappings, \( (f, f') < \mathcal{V} \) means that \( f \) and \( f' \) are \( \mathcal{V} \)-near mappings, i.e., for each \( x \in X \) there is a \( V \in \mathcal{V} \) such that \( f(x), f'(x) \in V \).

\[ \begin{align*}
(A1) & \quad (\forall a_2 \geq a_1 \geq a)(p_{aa}, p_{a_1a_2}, p_{aa_2}) < \mathcal{U}_a. \\
(A2) & \quad (\forall a \in A)(\forall \mathcal{U} \in \text{Cov}(X_a))(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a')(p_{aa}, p_{a_1a_2}, p_{aa_2}) < \mathcal{U}. 
\end{align*} \]
(A3) \((\forall a \in A)(\forall \mathcal{U} \in \text{Cov}(X_a))(\exists a' \geq a)(\forall a'' \geq a') \mathcal{U}_{a''} < p_{aa'}^{-1}(\mathcal{U} = \{p_{aa'}^{-1}(U) : U \in \mathcal{U}\}).\)

These approximate inverse systems are noncommutative inverse systems. They have many applications in dimension theory and shape theory (see [2, 5-7, 9-11]).

In this note we investigate the relation between conditions (A1)-(A3) and (A1)*-(A3)*. Our purpose is the following Theorem 1.

**Theorem 1.** Let \((A, \leq)\) be a directed preordered cofinite infinite set; \(X_a, a \in A\), be a compact metric space; \(p_{aa'} : X_a \to X_{a'}, a \leq a',\) be a mapping; and \(p_{aa}, a \in A,\) be the identity mapping. Then the following conditions are equivalent:

(A) There are coverings \(\mathcal{U}_a \in \text{Cov}(X_a), a \in A,\) such that \(\mathcal{U} = \{X_a, \mathcal{U}_a, p_{aa}, A\}\) satisfies (A1)-(A3).

(B) There are metrics \(d_a\) on \(X_a, a \in A,\) which induce the topology of \(X_a,\) such that \(\mathcal{U} = \{(X_a, d_a), \varepsilon_a, p_{aa}, A\}\) satisfies (A1)*-(A3)*.

(C) For any real numbers \(\varepsilon_a > 0, a \in A,\) there are metrics \(d_a\) on \(X_a, a \in A,\) which induce the topology of \(X_a,\) such that \(\mathcal{U} = \{(X_a, d_a), \varepsilon_a, p_{aa}, A\}\) satisfies (A1)*-(A3)*.

(D) There are real numbers \(\varepsilon_a > 0\) and metrics \(d_a\) on \(X_a, a \in A,\) which induce the topology of \(X_a,\) such that \(\mathcal{U} = \{(X_a, d_a), \varepsilon_a, p_{aa}, A\}\) satisfies (A1)*-(A3)*.

For our proof we need some lemmas. Let \(\mathcal{U}, \mathcal{V} \in \text{Cov}(X).\) For any subset \(K\) of \(X,\) we put \(st(K, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap K \neq \emptyset\}.\) When \(K = \{x\}\) is a singleton set, \(st(x, \mathcal{U})\) denotes \(st(\{x\}, \mathcal{U}).\) Let \(st \mathcal{U}\) be a covering \(\{st(U, \mathcal{U}) : U \in \mathcal{U}\}\) of \(X.\) Inductively, we define \(st^{n+1} \mathcal{U} = st(st^n \mathcal{U})\) and \(st^0 \mathcal{U} = \mathcal{U}\) for each integer \(n.\) We say \(st^n \mathcal{U}\) is the \(n\)th star covering of \(\mathcal{U}.\) When \(st^n \mathcal{U} < \mathcal{U}\), we say \(\mathcal{U}\) is an \(n\)-refinement of \(\mathcal{U}.\) Note that an open covering \(\mathcal{W}\) of \(X\) is normal provided there is a sequence of open coverings \(\mathcal{W}_i, i = 1, 2, \ldots,\) of \(X\) such that \(st \mathcal{W}_{i+1} < \mathcal{W}_i\) and \(\mathcal{W}_1 = \mathcal{W}.\) We call such a sequence a normal sequence of \(\mathcal{W}.\) Let \(\mathcal{W}^\Delta\) be a normal covering \(\{st(x, \mathcal{U}) : x \in X\}\) of \(X.\) Clearly \(\mathcal{W}^\Delta < st \mathcal{U}.\) Note that any open covering of a compact Hausdorff space is normal (see [8]). We can easily show the following:

**Lemma 2.** If \(\mathcal{U} = \{X_a, \mathcal{U}_a, p_{aa}, A\}\) satisfies (A1)-(A3), then \(st^n \mathcal{U} = \{X_a, \mathcal{U}_a, p_{aa}, A\}\) satisfies (A1)-(A3) for each integer \(n.\)

Let \((X, d)\) be a compact metric space. For any \(\varepsilon > 0,\) let \(S_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}\) for any \(x \in X\) and let \(\mathcal{S}_d(\varepsilon) = \{S_d(x, \varepsilon) : x \in X\}\).

**Lemma 3.** Let \((X, d)\) be a compact metric space. For any \(\mathcal{U} \in \text{Cov}(X),\) there exists a metric \(d^*\) on \(X\) satisfying

(i) \(d^*\) induces the topology of \(X,\)

(ii) \(\mathcal{U} < \mathcal{S}_d(2^{-5}) < st^4 \mathcal{U} < \mathcal{S}_d(2^{-1}) < st^8 \mathcal{U}.\)

**Proof.** Since \(d\) is a metric on \(X,\) \(\{\mathcal{S}_d(2^{-n}) : n = 1, 2, \ldots\}\) generates a uniformity \(\mu\) of \(X.\) Clearly, \(\text{Cov}(X)\) is also a uniformity of \(X.\) Since \(X\) is compact, we have the unique uniformity of \(X.\) Thus

(1) \(\mu = \text{Cov}(X).\)
Take any $\mathcal{U} \in \text{Cov}(X)$. Since $\mathcal{U}$ is normal, we have a sequence of open coverings $\mathcal{U}_i$, $i = 1, 2, \ldots$, of $X$ such that

(2) $\mathcal{U}_1 = \mathcal{U}$ and $\mathcal{U}_i > \text{st}\mathcal{U}_{i+1}$ for each integer $i$.

By (1) and (2) inductively it is easy to make a sequence of open coverings $\mathcal{V}_i$, $i = 1, 2, \ldots$, of $X$ such that

(3) $\mathcal{U}_i > \mathcal{V}_i$ and $\mathcal{I}_d(2^{-i}) > \mathcal{V}_i$ for each $i$,

(4) $\mathcal{V}_i > \text{st}\mathcal{V}_{i+1}$ for each $i$.

By (3) we have that

(5) $\{\text{st}(x, \mathcal{V}_i): i = 1, 2, \ldots\}$ is a neighborhood base of $x \in X$.

Now, let $\mathcal{W}_1 = \text{st}\mathcal{U}$, $\mathcal{W}_2 = \text{st}\mathcal{U}$, $\mathcal{W}_i = \text{st}\mathcal{U}_{8-i}$, $\ldots$, $\mathcal{W}_7 = \text{st}\mathcal{U}$, $\mathcal{W}_8 = \mathcal{U}$, $\mathcal{W}_9 = \mathcal{V}_2$, $\mathcal{V}_{10} = \mathcal{V}_3$, $\ldots$, $\mathcal{W}_j = \mathcal{V}_{j-7}$, $\ldots$. Thus by (4) and (5) we have that

(6) $\mathcal{W}_i > \text{st}\mathcal{W}_{i+1}$ for each $i$,

(7) $\{\text{st}(x, \mathcal{W}_i): i = 1, 2, \ldots\}$ is a neighborhood base of $x \in X$.

By (6), (7), and the proofs of 2-16 Theorem and 2-18 Corollary of [8, pp. 13-15], there is a metric $d^*$ on $X$ such that

(8) $d^*$ induces the topology of $X$,

(9) $\mathcal{W}_i < \mathcal{W}_i^\Delta < \mathcal{I}_d(2^{-i}) < \mathcal{W}_i^\Delta$ for each $i$.

Since $\mathcal{W}_i < \mathcal{W}_i^\Delta$ and $\mathcal{W}_i^\Delta < \text{st}\mathcal{W}_i$, by (9) for $i = 1, 5$ we have condition (ii). (8) means condition (i). Hence we have Lemma 3.

Proof of Theorem 1. First, we show (A) $\rightarrow$ (B). We assume condition (A) and take any $a \in A$. By Lemma 3 there is a metric $d_a^*$ on $X_a$ such that

(1) $d_a^*$ induces the topology of $X_a$,

(2) $\mathcal{U}_a < \mathcal{I}_d^a(2^{-5}) < \text{st}\mathcal{U}_a < \mathcal{I}_d^a(2^{-1}) < \text{st}\mathcal{U}_a$.

Let $d_a^**(x, x') = 2^4d_a^*(x, x')$ for $x, x' \in X_a$. Thus $d_a^**$ is a metric and by (1) we have

(3) $d_a^**$ induces the topology of $X_a$.

We show that $\mathcal{H}^** = \{(X_a, d_a^**), e_a, p_{aa'}, A\}$, $e_a = 1$ for $a \in A$, satisfies (A1)*$\rightarrow$*(A3)*. We consider (A1)* for $\mathcal{H}^**$. Take any $a_2 \geq a_1 \geq a$. By (A1) for $\mathcal{H}$, $(p_{aa}, p_{aa_1}, p_{aa_2}) < \mathcal{U}_a$. By (2) $d_a^*(p_{aa}, p_{aa_1}, p_{aa_2}) < 2 \cdot 2^{-5} = 2^{-4}$. Thus $d_a^**(p_{aa}, p_{aa_1}, p_{aa_2}) < 1 = e_a$. This means condition (A1)* for $\mathcal{H}^**$.

We consider (A2)* for $\mathcal{H}^**$. Take any $a \in A$ and any $\eta > 0$. We apply (A2) for $\mathcal{H}$ to $a$ and $\mathcal{I}_d^a(\eta/2)$. Thus there exists an $a' \geq a$ such that for each $a_2 \geq a_1 \geq a'$, $(p_{aa}, p_{aa_1}, p_{aa_2}) < \mathcal{I}_d^a(\eta/2)$. This means that $d_a^**(p_{aa}, p_{aa_1}, p_{aa_2}) < \eta$. Thus we have condition (A2)* for $\mathcal{H}^**$.

We consider (A3)* for $\mathcal{H}^**$. Take any $a \in A$ and any $\eta > 0$. By the assumption, $\mathcal{H}$ satisfies (A3). Thus by Lemma 2, $\text{st}\mathcal{H}$ also satisfies (A3). By applying (A3) for $\text{st}\mathcal{H}$ there is an $a' \geq a$ such that for any $a'' \geq a'$

(4) $p_{aa''}(\mathcal{I}_d^a(\eta/2)) > \text{st}\mathcal{H}$. 

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Take any $a'' \geq a'$ and any points $x, x' \in X_{a''}$ such that $d^{*\ast}_{a''}(x, x') \leq \epsilon_{a''} = 1$. Since $d^{*\ast}_{a''}(x, x') \leq 2^{-4}, x, x' \in S^{*\ast}_{a''}(x, 2^{-1})$. Thus by (2) $x, x' \in S^{*\ast}_{d^{*\ast}_{a''}}(x, 2^{-1}) \subset U$ for some $U \in S^{*\ast}_{a''}$. By (4) $U \subset p^{-1}_{a''}(S^{*\ast}_{d^{*\ast}}(z, \eta/2))$ for some $z \in X_a$. Then $p_{a''}(x), p_{a''}(x') \in p_{a''}(U) \subset S^{*\ast}_{d^{*\ast}}(z, \eta/2)$, and hence $d^{*\ast}_{a''}(p_{a''}(x), p_{a''}(x')) < \eta$. This means condition (A3)$^*$ for $\mathcal{R}^{**}$. Therefore we have (A) $\implies$ (B).

We show (B) $\implies$ (C). We may assume that $\mathcal{R} = \{(X_a, d_a), k_a, p_{a''}, A\}, k_a = 1$ for $a \in A$, satisfies (A1)$^*$--(A3)$^*$. Take any real numbers $\epsilon_a > 0, a \in A$. We put $d^*_a(x, x') = \epsilon_ad_a(x, x')$ for $x, x' \in X_a$ and $a \in A$. Clearly $d^*_a$ is a metric on $X_a$, and it is not difficult to show that $\mathcal{R}^{**} = \{(X_a, d^*_a), \epsilon_a, p_{a''}, A\}$ satisfies (A1)$^*$--(A3)$^*$. Therefore we have (B) $\implies$ (C).

Clearly, we have (C) $\implies$ (D) and (D) $\implies$ (A) is Theorem 1 of [3]. Hence, we complete the proof of Theorem 1.

**Remark 4.** We consider the following condition:

(E) For any metric $d_a$ on $X_a$ which induces the topology of $X_a$, $a \in A$, there are real numbers $\epsilon_a > 0, a \in A$, such that $\mathcal{R} = \{(X_a, d_a), \epsilon_a, p_{a''}, A\}$ satisfies (A1)$^*$--(A3)$^*$.

Clearly (E) $\implies$ (D). However, in general, (D) $\implies$ (E) does not hold because Example 1 of [3] satisfies (A) but not (E).

**References**


**DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, UNIVERSITY OF YAMAGUCHI, YAMAGUCHI CITY, 753 JAPAN**

**E-mail address:** f003000@ainet.ad.jp