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NUMERICAL MESHES AND COVERING MESHES OF APPROXIMATE INVERSE SYSTEMS OF COMPACTA

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ABSTRACT. Mardešić and Rubin (1989) introduced approximate inverse systems of metric compacta by the conditions (A1)*–(A3)*. Mardešić and Watanabe (1988) introduced approximate inverse systems of topological spaces by the conditions (A1)–(A3). In this note we show that any approximate inverse system of metric compacta satisfies (A1)–(A3) if and only if it satisfies (A1)*–(A3)* for some matrices (see Theorem 1).

S. Mardešić and L. Rubin [1] introduced the notion of approximate inverse system $\mathscr{X} = \{(X_a, d_a), \varepsilon_a, p_{aa'}, A\}$ of metric compacta. Hence, $(A, \leq)$ is a directed preordered infinite set. (X_a, d_a) is a compactum endowed with a metric d_a, and $p_{aa'}: X_{a'} \to X_a$ is a mapping defined whenever $a \leq a'$ and is such that p_{aa} is the identity mapping. The real numbers $\varepsilon_a > 0$, $a \in A$, are called numerical meshes. We require the following conditions:

(A1)* $(\forall a_2 \geq a_1 \geq a) d_a(p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leq \varepsilon_a$.
(A2)* $(\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a') d_a(p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leq \eta$.
(A3)* $(\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a'' \geq a')(\forall x, x' \in X_{a''}) d_{a''}(x, x') \leq \varepsilon_{a''}$ implies $d_a(p_{aa''}(x), p_{aa''}(x')) \leq \eta$.

In [6] S. Mardešić and T. Watanabe introduced the notion of an approximate inverse system $\mathscr{X} = \{X_a, \mathscr{U}_a, p_{aa'}, A\}$ of topological spaces. Here A is a directed preordered infinite set, $\mathscr{U}_a$ is a normal open covering of X_a, $a \in A$, also called a mesh, and $p_{aa'}: X_{a'} \to X_a$ is a mapping defined whenever $a \leq a'$ and is such that p_{aa} is the identity mapping. We require three conditions (A1)–(A3), which are natural analogues of conditions (A1)*–(A3)*. Before we state these conditions, let us denote that $\mathrm{Cov}(X)$ is the set of all normal open coverings of a space X. For $\mathscr{U}, \mathscr{V} \in \mathrm{Cov}(X)$, $\mathscr{V} < \mathscr{U}$ means that $\mathscr{V}$ refines $\mathscr{U}$. If $\mathscr{V} \in \mathrm{Cov}(Y)$ and $f, f': X \to Y$ are mappings, $(f, f') < \mathscr{V}$ means that f and f' are $\mathscr{V}$-near mappings, i.e., for each $x \in X$ there is a $V \in \mathscr{V}$ such that $f(x), f'(x) \in V$.

(A1) $(\forall a_2 \geq a_1 \geq a)(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \mathscr{U}_a$.
(A2) $(\forall a \in A)(\forall \mathscr{U} \in \mathrm{Cov}(X_a))(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a')(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \mathscr{U}$.

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(A3) $(\forall a \in A)(\forall \mathscr{U} \in \operatorname{Cov}(X_a))(\exists a' \geq a)(\forall a'' \geq a')\ \mathscr{U}_{a''} < p_{aa''}^{-1}(\mathscr{U}) = \{p_{aa''}^{-1}(U) : U \in \mathscr{U}\}$.

These approximate inverse systems are noncommutative inverse systems. They have many applications in dimension theory and shape theory (see [2, 5–7, 9–11]).

In this note we investigate the relation between conditions (A1)–(A3) and (A1)*–(A3)*. Our purpose is the following Theorem 1.

Theorem 1. *Let $(A, \leq)$ be a directed preordered cofinite infinite set; X_a, $a \in A$, be a compact metric space; $p_{aa'} : X_{a'} \to X_a$, $a \leq a'$, be a mapping; and p_{aa}, $a \in A$, be the identity mapping. Then the following conditions are equivalent:*

(A) *There are coverings $\mathscr{U}_a \in \operatorname{Cov}(X_a)$, $a \in A$, such that $\mathscr{X} = \{X_a, \mathscr{U}_a, p_{aa'}, A\}$ satisfies* (A1)–(A3).

(B) *There are metrics d_a on X_a, $a \in A$, which induce the topology of X_a, such that $\mathscr{X} = \{(X_a, d_a), \varepsilon_a, p_{aa'}, A\}$, $\varepsilon_a = 1$ for $a \in A$, satisfies* (A1)*–(A3)*.

(C) *For any real numbers $\varepsilon_a > 0$, $a \in A$, there are metrics d_a on X_a, $a \in A$, which induce the topology of X_a, such that $\mathscr{X} = \{(X_a, d_a), \varepsilon_a, p_{aa'}, A\}$ satisfies* (A1)*–(A3)*.

(D) *There are real numbers $\varepsilon_a > 0$ and metrics d_a on X_a, $a \in A$, which induce the topology of X_a, such that $\mathscr{X} = \{(X_a, d_a), \varepsilon_a, p_{aa'}, A\}$ satisfies* (A1)*–(A3)*.

For our proof we need some lemmas. Let $\mathscr{U}, \mathscr{V} \in \operatorname{Cov}(X)$. For any subset K of X, we put $\operatorname{st}(K, \mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap K \neq \varnothing\}$. When $K = \{x\}$ is a singleton set, $\operatorname{st}(x, \mathscr{U})$ denotes $\operatorname{st}(\{x\}, \mathscr{U})$. Let $\operatorname{st}\mathscr{U}$ be a covering $\{\operatorname{st}(U, \mathscr{U}) : U \in \mathscr{U}\}$ of X. Inductively, we define $\operatorname{st}^{n+1}\mathscr{U} = \operatorname{st}(\operatorname{st}^n\mathscr{U})$ and $\operatorname{st}^0\mathscr{U} = \mathscr{U}$ for each integer n. We say $\operatorname{st}^n\mathscr{U}$ is the nth star covering of $\mathscr{U}$. When $\operatorname{st}^n\mathscr{U} < \mathscr{V}$, we say $\mathscr{U}$ is an n-refinement of $\mathscr{V}$. Note that an open covering $\mathscr{W}$ of X is norma provided there is a sequence of open coverings $\mathscr{W}_i$, $i = 1, 2, \dots$, of X such that $\operatorname{st}\mathscr{W}_{i+1} < \mathscr{W}_i$ and $\mathscr{W}_1 = \mathscr{W}$. We call such a sequence a normal sequence of $\mathscr{W}$. Let $\mathscr{U}^\Delta$ be a normal covering $\{\operatorname{st}(x, \mathscr{U}) : x \in X\}$ of X. Clearly $\mathscr{U}^\Delta < \operatorname{st}\mathscr{U}$. Note that any open covering of a compact Hausdorff space is normal (see [8]). We can easily show the following:

Lemma 2. *If $\mathscr{X} = \{X_a, \mathscr{U}_a, p_{aa'}, A\}$ satisfies* (A1)–(A3), *then $\operatorname{st}^n\mathscr{X} = \{X_a, \operatorname{st}^n\mathscr{U}_a, p_{aa'}, A\}$ satisfies* (A1)–(A3) *for each integer n.*

Let (X, d) be a compact metric space. For any $\varepsilon > 0$, let $S_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for any $x \in X$ and let $\mathscr{S}_d(\varepsilon) = \{S_d(x, \varepsilon) : x \in X\}$.

Lemma 3. *Let (X, d) be a compact metric space. For any $\mathscr{U} \in \operatorname{Cov}(X)$, there exists a metric d^* on X satisfying*

(i) *d^* induces the topology of X,*

(ii) $\mathscr{U} < \mathscr{S}_{d^*}(2^{-5}) < \operatorname{st}^4\mathscr{U} < \mathscr{S}_{d^*}(2^{-1}) < \operatorname{st}^8\mathscr{U}$.

Proof. Since d is a metric on X, $\{\mathscr{S}_d(2^{-n}) : n = 1, 2, \dots\}$ generates a uniformity μ of X. Clearly, $\operatorname{Cov}(X)$ is also a uniformity of X. Since X is compact, we have the unique uniformity of X. Thus

$$\mu = \operatorname{Cov}(X). \tag{1}$$

Take any $\mathscr{U} \in \operatorname{Cov}(X)$. Since $\mathscr{U}$ is normal, we have a sequence of open coverings $\mathscr{U}_i$, $i = 1, 2, \ldots$, of X such that

$$\text{(2)} \qquad \mathscr{U}_1 = \mathscr{U} \quad \text{and} \quad \mathscr{U}_i > \operatorname{st}\mathscr{U}_{i+1} \quad \text{for each integer } i.$$

By (1) and (2) inductively it is easy to make a sequence of open coverings $\mathscr{V}_i$, $i = 1, 2, \ldots$, of X such that

$$\text{(3)} \qquad \mathscr{U}_i > \mathscr{V}_i \quad \text{and} \quad \mathscr{S}_d(2^{-i}) > \mathscr{V}_i \quad \text{for each } i,$$

$$\text{(4)} \qquad \mathscr{V}_i > \operatorname{st}\mathscr{V}_{i+1} \quad \text{for each } i.$$

By (3) we have that

$$\text{(5)} \qquad \{\operatorname{st}(x, \mathscr{V}_i) : i = 1, 2, \ldots\} \text{ is a neighborhood base of } x \in X.$$

Now, let $\mathscr{W}_1 = \operatorname{st}^7\mathscr{U}$, $\mathscr{W}_2 = \operatorname{st}^6\mathscr{U}, \ldots, \mathscr{W}_i = \operatorname{st}^{8-i}\mathscr{U}, \ldots, \mathscr{W}_7 = \operatorname{st}\mathscr{U}$, $\mathscr{W}_8 = \mathscr{U}$, $\mathscr{W}_9 = \mathscr{V}_2$, $\mathscr{W}_{10} = \mathscr{V}_3, \ldots, \mathscr{W}_j = \mathscr{V}_{j-7}, \ldots$. Thus by (4) and (5) we have that

$$\text{(6)} \qquad \mathscr{W}_i > \operatorname{st}\mathscr{W}_{i+1} \quad \text{for each } i,$$

$$\text{(7)} \qquad \{\operatorname{st}(x, \mathscr{W}_i) : i = 1, 2, \ldots\} \text{ is a neighborhood base of } x \in X.$$

By (6), (7), and the proofs of 2–16 Theorem and 2–18 Corollary of [8, pp. 13–15], there is a metric d^* on X such that

$$\text{(8)} \qquad d^* \text{ induces the topology of } X,$$

$$\text{(9)} \qquad \mathscr{W}_{i+3}^{\Delta} < \mathscr{S}_{d^*}(2^{-i}) < \mathscr{W}_i^{\Delta} \quad \text{for each } i.$$

Since $\mathscr{W}_i < \mathscr{W}_i^{\Delta}$ and $\mathscr{W}_i^{\Delta} < \operatorname{st}\mathscr{W}_i$, by (9) for $i = 1, 5$ we have condition (ii). (8) means condition (i). Hence we have Lemma 3.

Proof of Theorem 1. First, we show (A) $\to$ (B). We assume condition (A) and take any $a \in A$. By Lemma 3 there is a metric d_a^* on X_a such that

$$\text{(1)} \qquad d_a^* \text{ induces the topology of } X_a,$$

$$\text{(2)} \qquad \mathscr{U}_a < \mathscr{S}_{d_a^*}(2^{-5}) < \operatorname{st}^4\mathscr{U}_a < \mathscr{S}_{d_a^*}(2^{-1}) < \operatorname{st}^8\mathscr{U}_a.$$

Let $d_a^{**}(x, x') = 2^4 d_a^*(x, x')$ for x, $x' \in X_a$. Thus d_a^{**} is a metric and by (1) we have

$$\text{(3)} \qquad d_a^{**} \text{ induces the topology of } X_a.$$

We show that $\mathscr{X}^{**} = \{(X_a, d_a^{**}), \varepsilon_a, p_{aa'}, A\}$, $\varepsilon_a = 1$ for $a \in A$, satisfies (A1)*–(A3)*. We consider (A1)* for $\mathscr{X}^{**}$. Take any $a_2 \geq a_1 \geq a$. By (A1) for $\mathscr{X}$, $(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \mathscr{U}_a$. By (2) $d_a^*(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < 2 \cdot 2^{-5} = 2^{-4}$. Thus $d_a^{**}(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < 1 = \varepsilon_a$. This means condition (A1)* for $\mathscr{X}^{**}$.

We consider (A2)* for $\mathscr{X}^{**}$. Take any $a \in A$ and any $\eta > 0$. We apply (A2) for $\mathscr{X}$ to a and $\mathscr{S}_{d_a^{**}}(\eta/2)$. Thus there exists an $a' \geq a$ such that for each $a_2 \geq a_1 \geq a'$, $(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \mathscr{S}_{d_a^{**}}(\eta/2)$. This means that $d_a^{**}(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \eta$. Thus we have condition (A2)* for $\mathscr{X}^{**}$.

We consider (A3)* for $\mathscr{X}^{**}$. Take any $a \in A$ and any $\eta > 0$. By the assumption, $\mathscr{X}$ satisfies (A3). Thus by Lemma 2, $\operatorname{st}^8\mathscr{X}$ also satisfies (A3). By applying (A3) for $\operatorname{st}^8\mathscr{X}$ there is an $a' \geq a$ such that for any $a'' \geq a'$

$$\text{(4)} \qquad p_{aa''}^{-1}(\mathscr{S}_{d_a^{**}}(\eta/2)) > \operatorname{st}^8\mathscr{U}_{a''}.$$

Take any $a'' \geq a'$ and any points x, $x' \in X_{a''}$ such that $d_{a''}^{**}(x, x') \leq \varepsilon_{a''} = 1$. Since $d_{a''}^{*}(x, x') \leq 2^{-4}$, x, $x' \in S_{d_{a''}^{*}}(x, 2^{-1})$. Thus by (2) x, $x' \in S_{d_{a''}^{*}}(x, 2^{-1}) \subset U$ for some $U \in \operatorname{st}^8 \mathscr{U}_{a''}$. By (4) $U \subset p_{aa''}^{-1}(S_{d_a^{**}}(z, \eta/2))$ for some $z \in X_a$. Then $p_{aa''}(x)$, $p_{aa''}(x') \in p_{aa''}(U) \subset S_{d_a^{**}}(z, \eta/2)$, and hence $d_a^{**}(p_{aa''}(x), p_{aa''}(x')) < \eta$. This means condition (A3)* for $\mathscr{X}^{**}$. Therefore we have (A) $\rightarrow$ (B).

We show (B) $\rightarrow$ (C). We may assume that $\mathscr{X} = \{(X_a, d_a), k_a, p_{aa'}, A\}$, $k_a = 1$ for $a \in A$, satisfies (A1)*–(A3)*. Take any real numbers $\varepsilon_a > 0$, $a \in A$. We put $d_a^*(x, x') = \varepsilon_a d_a(x, x')$ for x, $x' \in X_a$ and $a \in A$. Clearly d_a^* is a metric on X_a, and it is not difficult to show that $\mathscr{X}^* = \{(X_a, d_a^*), \varepsilon_a, p_{aa'}, A\}$ satisfies (A1)*–(A3)*. Therefore we have (B) $\rightarrow$ (C).

Clearly, we have (C) $\rightarrow$ (D) and (D) $\rightarrow$ (A) is Theorem 1 of [3]. Hence, we complete the proof of Theorem 1.

Remark 4. We consider the following condition:

(E) For any metric d_a on X_a which induces the topology of X_a, $a \in A$, there are real numbers $\varepsilon_a > 0$, $a \in A$, such that $\mathscr{X} = \{(X_a, d_a), \varepsilon_a, p_{aa'}, A\}$ satisfies (A1)*–(A3)*.

Clearly (E) $\rightarrow$ (D). However, in general, (D) $\rightarrow$ (E) does not hold because Example 1 of [3] satisfies (A) but not (E).

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