PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

A PROPER G_a ACTION ON C^5 WHICH IS NOT LOCALLY TRIVIAL

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. The quotient of a proper holomorphic G_a action on $\mathbf{C}^n$ is known to carry the structure of a complex analytic manifold, and in the case of a rational algebraic action, the geometric quotient exists as an algebraic space. An example is given of a proper rational algebraic action for which the quotient is not a variety, and therefore the action is not locally trivial in the Zariski topology.

1. INTRODUCTION

Let G_a denote the additive group of complex numbers, X a variety over $\mathbf{C}$, and $\sigma : G_a \times X \to X$ a rational action of G_a on X. The action is said to admit an equivariant trivialization if X is (G_a) equivariantly isomorphic to $Y \times \mathbf{C}$, with the group action fixing the first coordinate and acting by addition on the second. In that case, the affine variety Y is a geometric quotient. The action is said to be locally trivial (in the Zariski topology) if X is covered by G_a stable (affine) open subsets on each of which the action admits a equivariant trivialization.

If $(\sigma, \mathrm{id}) : G_a \times X \to X \times X$ is a proper morphism of varieties, we say that σ is a proper action. A proper action of G_a on $X = \mathbf{C}^n$ is locally trivial if $\mathbf{C}[X]$ is a flat extension of its subring of G_a invariants and equivariantly trivial if the extension is faithfully flat [1]. Moreover, under those conditions the geometric exists as a quasiaffine variety.

These results were used to show that all proper fixed point free G_a actions on $\mathbf{C}^3$ admit equivariant trivializations. Fauntleroy [4] has shown that locally trivial G_a actions are necessarily proper. Smith (see [10]) and Winkelmann [12] have given examples of fixed point free actions on $\mathbf{C}^4$ for which the space of orbits is not Hausdorff in the quotient topology induced from the complex topology on $\mathbf{C}^4$. These examples are clearly nonproper, and admit no geometric quotient. In the same paper, Winkelmann has given an example of a proper action on $\mathbf{C}^5$ which is locally trivial in the Zariski topology but does not admit an equivariant trivialization.

Received by the editors March 15, 1993.
1991 *Mathematics Subject Classification*. Primary 14L30; Secondary 20G20, 12F20.
The second author was supported in part by NSA Grant MDA904-H-3072.

The second section discusses some features of proper rational algebraic G_a actions on $\mathbf{C}^n$, giving some indication of how close proper actions are to being locally trivial. In the last section an example is given of a proper action on complex affine five space which is not locally trivial. It must be noted that Fauntleroy [4] has shown that if distinct orbits of a proper G_a action on a quasifactorial variety can be separated by invariant rational functions (i.e., if the action is properly stable), then the action is locally trivial. It is the stability that is lacking by our example. An assertion which would imply that any proper, fixed point free G_a action on a normal variety is locally trivial and admits a quasiprojective quotient appears in a paper of Magid and Fauntleroy [5], and the source of the error is pointed out in [4]. The example here indicates that no such general result is possible.

2. Proper algebraic actions on $\mathbf{C}^n$

Let $\sigma : G_a \times X \to X$ be a rational action of G_a on $X = \mathbf{C}^n$, $\hat{\sigma} : \mathbf{C}[X] \to \mathbf{C}[X, t]$ the induced map on coordinate rings, and $\tilde{\sigma}$ the morphism $G_a \times X \to X \times X$ given by $(t, x) \mapsto (x, \sigma(t, x))$. Differentiating $\hat{\sigma}$ yields a locally nilpotent derivation δ of $\mathbf{C}[X]$:

$$\delta(P) = \left.\frac{\hat{\sigma}(P) - P}{t}\right|_{t=0}, \qquad \hat{\sigma} = \exp(t\delta).$$

Every $\hat{\sigma}$, hence every rational G_a action, arises as the exponential of a locally nilpotent derivation. It should be noted that the ring of invariants of the G_a action is identical to the kernel of δ and that the fixed point set for the action is the set of common zeros of $\{\delta x_i : 1 \le i \le n\}$ where x_i, $1 \le i \le n$, are coordinates on X.

Properness of the action is expressed in terms of coordinate rings: $\tilde{\sigma}$ induces a $\mathbf{C}$-algebra homomorphism $\overline{\sigma} : \mathbf{C}[X \times X] \to \mathbf{C}[X \times G_a] \cong \mathbf{C}[X, t]$. It was proved in [1] that σ is proper if and only if $\overline{\sigma}$ is surjective, and locally trivial if and only if $\mathrm{im}(\delta) \cap \mathbf{C}[X]^{G_a}$ generates the unit ideal in $\mathbf{C}[X]$. In the latter case the action admits a quasiaffine geometric quotient. An easy consequence of the surjectivity of $\overline{\sigma}$ is the absence of fixed points. Moreover, for proper G_a actions we have shown that the action is locally trivial if and only if the ring extension $\mathbf{C}[X]^{G_a} \hookrightarrow \mathbf{C}[X]$ is flat, and equivariantly trivial if and only if the extension is faithfully flat [1]. If there is a slice for the action, i.e. $s \in \mathbf{C}[X]$ with $\hat{\sigma}(s) = s + t$ (equivalently, $\delta(s) = 1$), then clearly $\overline{\sigma}$ is surjective. Moreover, if a slice exists, then the action is equivariantly trivial, i.e., $X \cong X/G_a \times G_a$.

For the remainder of this section assume that σ is a proper action of G_a on $X = \mathbf{C}^n$, so that for some $P \in \mathbf{C}[X \times X] = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n]$, we have

$$(*) \qquad t = \overline{\sigma}P(x_1, \dots, x_n, y_1, \dots, y_n) = P(x_1, \dots, x_n, \hat{\sigma}x_1, \dots, \hat{\sigma}x_n).$$

For $a = (a_1, \dots, a_n) \in X$ define $P_a = P(a_1, \dots, a_n, x_1, \dots, x_n) \in \mathbf{C}[X]$. Clearly P_a provides an algebraic isomorphism from the orbit of a to $\mathbf{C}$. In particular, each orbit is a complete intersection [7]. The following remarks show how close P_a is to a slice.

Differentiate $(*)$ with respect to t, using the fact that $\hat{\sigma} = \exp(t\delta)$, to obtain

$$1 = \sum_{i=1}^{n} \frac{\partial P_a}{\partial x_i} \delta x_i(a_i).$$

The definition of the tangent space to the zero set of P_a at a (e.g., [9, p. 73]) shows that the zero set of P_a intersects the orbit of a transversally at a [9, p. 82], and it is clear that a is the unique point in the intersection.

If $\{h_1, \dots, h_{n-1}\}$ is a minimal generating set for the ideal of the orbit of a, then $h_1, \dots, h_{n-1}, P_a$ are local parameters at a in X. Moreover, since a is simple on the zero set of P_a, only one component, say $Z_a =$ the zero set of q, passes through it. The h_i need not be invariants, however. If the orbit of a can be determined by the vanishing of invariant rational functions g_i, then on an affine neighborhood U_a of a, $Z_a \cap U_a$ gives a local quotient for the action. Indeed, because Z_a intersects the orbit of a transversally, the local parameters g_i, q form a coordinate system for U_a at a. In particular, orbits on U_a are determined by values of the rational functions g_i.

3. A PROPER G_a ACTION ON $\mathbf{C}^5$ WHICH IS NOT LOCALLY TRIVIAL

The action is determined by the locally nilpotent derivation δ of $\mathbf{C}[x_1, x_2, y_1, y_2, z]$ given by

$$x_2 \overset{\delta}{\mapsto} x_1 \overset{\delta}{\mapsto} 0, \qquad y_2 \overset{\delta}{\mapsto} y_1 \overset{\delta}{\mapsto} 0, \qquad z \overset{\delta}{\mapsto} (1 + x_1 y_2^2).$$

To see that the action is proper, observe that $t = \hat{\sigma} z - z - y_2^2(\hat{\sigma} x_2 - x_2) - y_2(\hat{\sigma} x_2 - x_2)(\hat{\sigma} y_2 - y_2) - \frac{(\hat{\sigma} x_2 - x_2)(\hat{\sigma} y_2 - y_2)^2}{3}$.

We show that there are distinct orbits which are not separable by invariant rational functions. In fact, generators for the ring of invariants are explicitly given, from which it is clear that the ring of invariants is not regular. With aid of the computer algebra program MAPLE, it is easy to check that the action is unstable by checking the elements of a Gröbner basis for the kernel of $\hat{\sigma}$ against those of a Gröbner basis for the ideal of $\mathbf{C}[X, Y]$ generated by $\{c(X) - c(Y) : c \in \mathbf{C}[X]^{G_a}\}$. Instability clearly is due to the inability to separate orbits by G_a invariants; the use of computational methods is mentioned to emphasize their utility in problems such as these.

The ring of invariants is generated by the five polynomials $c_1 = x_1$, $c_2 = y_1$, and

$$c_3 = x_1 y_2 - x_2 y_1,$$
$$c_4 = 3 y_1 z - x_1 y_2^3 - 3 y_2,$$
$$c_5 = \frac{x_1^2 c_4 + c_3^3 + 3 x_1 c_3}{y_1}.$$

These generators were obtained by implementing a form of the algorithm in [11], easily extended to locally nilpotent, but not necessarily linear, derivations of polynomial rings. It should be noted that van den Essen has given a treatment of the algorithm, suitable for computer implementation, in [3]. Since the latter reference may not be easily accessible, we sketch the application to the example at hand, referring the reader to [11] for details.

It follows from [2, §2] that the ring of G_a invariants for the action extended to $\mathbf{C}[X, \frac{1}{y_1}]$ is generated by $\{\frac{1}{y_1}, c_1, c_2, c_3, c_4\}$. Indeed, $\frac{y_2}{y_1}$ is a slice for the extended action. We build a chain of subrings C_i of the ring of G_a invariants in $\mathbf{C}[X]$ as follows:

Set $C_1 = \mathbf{C}[c_1, c_2, c_3, c_4]$ and $\overline{C}_1 = C_1/(y_1)$. View $\overline{C}_1$ as the homomorphic image of the polynomial ring $\mathbf{C}[z_1, z_3, z_4]$ under the mapping which sends

z_i to the residue class of c_i. Since $\overline{C}_1$ is a two-dimensional domain, the kernel is a principal ideal, easily seen to be generated by $z_1^2 z_4 + z_3^3 + 3z_1 z_3$. In other words, $x_1^2 c_4 + c_3^3 + 3x_1 c_3$ lies in the ideal of C_1 generated by y_1. The invariant c_5 is obtained by dividing out the highest power $(= 1)$ of y_1 dividing $x_1^2 c_4 + c_3^3 + 3x_1 c_3$ and C_2 is defined to be $C_1[c_5]$.

Repeating this procedure with $\overline{C}_2 = C_2/(y_1)$, one finds that the residue class of c_5 is algebraically independent from the classes c_1, c_2, c_3, c_4. In particular, no new relations, and hence no new invariants, arise.

Thus the ring of invariants C is isomorphic to $\mathbf{C}[u_1, u_2, u_3, u_4, u_5]/\langle u_2 u_5 - u_1^2 u_4 - u_3^3 - 3u_1 u_3\rangle$ which is the coordinate ring of a variety with singularities at all points $p_\alpha = (0, 0, 0, \alpha, 0)$. If we denote $\operatorname{spec} C$ by Y and by π the morphism induced by the inclusion $C \hookrightarrow \mathbf{C}[\mathbf{X}]$, then we see that the fiber $\pi^{-1}(p_\alpha)$ is two dimensional, consisting of all points $(0, \beta, 0, \frac{-\alpha}{3}, \gamma)$. In particular, the extension $C \hookrightarrow \mathbf{C}[\mathbf{X}]$ is not flat [6, Theorem 15.1]. For the reader's convenience we make the following remark.

Remark 3.1. For an action locally trivial in the Zariski topology, the extension $C \hookrightarrow \mathbf{C}[\mathbf{X}]$ is necessarily flat.

Indeed, local triviality implies that there is an open cover of X by principal affine subsets $X_{f_i} \cong Y_i \times \mathbf{C}$, where $f_i \in C \cap \operatorname{image}(\delta)$. Clearly the projection morphism $X_{f_i} \to Y_i$ is flat. Since flatness is a local condition, the result follows.

Distinct orbits can, however, be separated by algebraic functions. By the method of Seshandri [8, Theorem 6.1], the G_a action extends to the normalization Z of X in a certain degree six Galois extension of $\mathbf{C}(X)$. The action on Z is locally trivial and admits a geometric quotient W, which is a variety (necessarily not quasiprojective by [8, p. 543]). If G denotes the Galois group of $\mathbf{C}(Z)/\mathbf{C}(X)$, then, as indicated in [4], the geometric quotient of X is the algebraic space W/G.

The affine variety Z is constructed as follows. Let S_1 be the hyperplane in X defined by $z = 0$ and S_2 the hyperplane $x_2 = y_2$. One checks that $X = \sigma(G_a \times S_1) \cup \sigma(G_a \times S_2)$. Denote $\mathbf{C}(G_a \times S_i)$ by K_i, observing that they are field extensions of $\mathbf{C}(X)$ since the U_i are dense in X. If $u_{11} = x_1$, $u_{12} = x_2$, $u_{13} = y_1$, and $u_{14} = y_2$ are coordinates on S, and t is the coordinate on G_a, then the field extension $\mathbf{C}(X) \hookrightarrow K_1$ is given by $x_1 \mapsto u_{11}$, $x_2 \mapsto u_{12} + tu_{12}$, $y_1 \mapsto u_{13}$, $y_2 \mapsto u_{13} + tu_{14}$, $z \mapsto t(1 + u_{11}u_{14}^2) + t^2 u_{11}u_{13}u_{14} + \frac{t^3}{3}u_{11}u_{13}^2$. Applying the invariants $c_1, \dots, c_5$ to these expressions shows that t satisfies a cubic polynomial over $C(X)$ and $K_1 = \mathbf{C}(X)(t)$.

A similar procedure shows that $K_2 = \mathbf{C}(X)$. Now Z is taken to be the normalization of X in the normal closure of K_1 over $\mathbf{C}(X)$. The slices on Z are given by $\frac{x_2}{x_1}$, $\frac{y_2}{y_1}$, and the three roots of the minimal polynomial for t in $\mathbf{C}(Z)$.

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