A COUNTER-EXAMPLE ON A QUASI-VARIATIONAL INEQUALITY WITHOUT LOWER SEMICONTINUITY ASSUMPTION

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Abstract. In this note we show that a recent result on quasi-variational inequalities, which seems to improve deeply some well-known results, is not correct.

In a very recent paper, N. H. Dien stated the following assertion.

Assertion A (Theorem 3 in [2]). Let $C$ be a nonempty compact and convex set in $\mathbb{R}^n$, $K: C \rightarrow 2^C$ a multifunction, and $\varphi: C \rightarrow \mathbb{R} \cup \{\pm \infty\}$, $f: C \rightarrow \mathbb{R}^n$, and $\eta: C \times C \rightarrow \mathbb{R}^n$ three (single-valued) functions. Assume that:

1. $K$ is upper semicontinuous with nonempty compact convex values;
2. $\varphi$ is lower semicontinuous (in the sense of single-valued maps) and convex;
3. $\eta$ and $f$ are continuous and $\langle f(x), \eta(x, x) \rangle \geq 0$ for all $x \in C$;
4. for each $x \in C$, the function $\langle f(x), \eta(\cdot, x) \rangle$ is convex on $C$.

Then there exists $\bar{x} \in C$ such that

1. $\bar{x} \in K(\bar{x})$ and $\langle f(\bar{x}), \eta(u, \bar{x}) \rangle + \varphi(u) - \varphi(\bar{x}) \geq 0$ for all $u \in K(\bar{x})$.

The aim of the present note is to show that Assertion A, in general, is false. Notice that the above statement, if true, would imply (taking $\eta(y, x) = y - x$ and $\varphi \equiv 0$) that the classical and celebrated Chan and Pang's result on quasi-variational inequalities (Corollary 3.1 in [1]) is still true (for single-valued $f$) without supposing the multifunction $K$ to be lower semicontinuous on $C$. The following simple example shows at once that Assertion A is incorrect and that the mentioned improvement of Chan and Pang's theorem is not possible.

Example. Take $C = [0, 1]$, $\varphi \equiv 0$, $\eta(y, x) = y - x$, $f(x) = 1$, and let the multifunction $K$ be defined by

$$K(x) = \begin{cases} \left[\frac{1}{2}, 1\right] & \text{if } x \in [0, \frac{1}{2}], \\
\left[0, 1\right] & \text{if } x = \frac{1}{2}, \\
\left[0, \frac{1}{2}\right] & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

The reader can easily verify that the graph of the multifunction $K$ is closed and then, since $[0, 1]$ is compact, $K$ is upper semicontinuous. Moreover, the other
assumptions of Assertion A are trivially satisfied. However, the multifunction $K$ has just one fixed point which is not a solution to the problem (1). Notice that such $K$ is not lower semicontinuous.

Remark 1. The mistake in the proof of Theorem 3 in [2] resides in the assertion that the marginal function $S$ is upper semicontinuous. To see this, let us consider the space $C$, the multifunction $K$, and the maps $f$, $\eta$, $\varphi$ as in the example. In this case, the multifunction $S$ becomes

$$S(x) = \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0, \frac{1}{2}], \\ \{0\} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

which is not upper semicontinuous.


REFERENCES


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