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REMARKS ON THE TOPOLOGY OF FOLDS

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ABSTRACT. We give some necessary conditions for a closed manifold to admit a smooth map into an Euclidean space with only fold singular points.

1. INTRODUCTION

Let $f: M^n \to \mathbf{R}^p$ $(n \geq p)$ be a smooth map of a closed n-dimensional manifold into the p-dimensional Euclidean space. We say that a singular point $q \in M$ of f is a *fold singular point of index* λ if f has the normal form as follows for some local coordinate systems around q and $f(q)$:

$$\begin{aligned} y_i \circ f &= x_i \qquad (1 \leq i \leq p-1), \\ y_p \circ f &= -x_p^2 - \cdots - x_{p+\lambda-1}^2 + x_{p+\lambda}^2 + \cdots + x_n^2, \end{aligned}$$

where $0 \leq \lambda \leq n-p+1$ is an integer. If $\lambda = 0$ or $n-p+1$, we say that q is a *definite fold singular point.* Our main results of this paper are as follows.

Theorem 1.1. *If a closed n-dimensional manifold* M^n *admits a smooth map* $f: M^n \to \mathbf{R}^p$ $(n \geq p)$ *with only definite fold singular points, then it is smoothly null-cobordant. If, in addition,* M^n *is oriented, it is null-cobordant in the oriented category.*

Theorem 1.2. *Let* M^n *be a closed n-dimensional manifold with odd Euler number. If* M^n *admits a smooth map* $f: M^n \to \mathbf{R}^p$ $(n \geq p)$ *with only fold singular points, then* $p = 1, 3,$ *or* 7.

Note that Theorem 1.1 has been proved in [7, Corollary 3.3] when $n > p$. Note also that the fact that p must be odd in Theorem 1.2 has been obtained in [8, Theorem 1].

2. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. If $n > p$, the theorem is proved in [7]. Furthermore, if $n = p$ and M^n is oriented, then M^n is stably parallelizable by [2] (see also [7, Proposition 7.4]), and hence it is null-cobordant in the oriented category, since

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all the characteristic classes vanish. Thus we assume that $n = p$ and M^n is nonorientable. In this case, by Corollary 7.5 of [7], the Stiefel-Whitney classes $w_i(M^n)$ vanish for all $i \geq 2$. Thus it suffices to prove the following.

Lemma 2.1. *Let M^n be a closed n-dimensional manifold whose Stiefel-Whitney classes $w_i(M^n)$ vanish for all $i \geq 2$. Then it is null-cobordant.*

Proof. Let $\overline{w}(M) = 1+\overline{w}_1(M)+\cdots+\overline{w}_n(M) \in H^*(M; \mathbf{Z}_2)$ be the dual Stiefel-Whitney class of M. By the hypothesis, we see easily that

$$\overline{w}(M) = 1 + w_1(M) + w_1(M)^2 + \cdots + w_1(M)^n$$

and, in particular, that

$$\overline{w}_n(M) = w_1(M)^n.$$

On the other hand, it is well known that $\overline{w}_n(M)$ always vanishes for every n-dimensional manifold M, since, by Whitney [10], it can be immersed into $\mathbf{R}^{2n-1}$ if $n \geq 2$. (See also [6, p. 136].) Thus we see that $w_1(M)^n = 0$ and, hence, that the corresponding Stiefel-Whitney number vanishes. By our hypothesis, all the other Stiefel-Whitney numbers also vanish. Thus M is null-cobordant. This completes the proof of Lemma 2.1 and, hence, of Theorem 1.1. □

Proof of Theorem 1.2. Let $S(f)$ be the singular set of the smooth map $f: M^n \to \mathbf{R}^p$; i.e., $S(f) = \{q \in M;\ \operatorname{rank} df_q < p\}$. It is easy to see, by the definition of a fold singular point, that $S(f)$ is a $(p-1)$-dimensional closed submanifold of M and that $f|S(f): S(f) \to \mathbf{R}^p$ is a codimension-1 immersion. On the other hand, by Fukuda [3] and our hypothesis, we see that the Euler number of $S(f)$ is odd. In particular, $S(f)$ is not null-cobordant. Then, by a theorem of Brown [1] (see also [5]), we see that $S(f)$ must be cobordant to $\mathbf{R}P^0$, $\mathbf{R}P^2$, or $\mathbf{R}P^6$. This shows that $p-1 = 0, 2$, or 6. This completes the proof of Theorem 1.2. □

Remark 2.2. Note that, for $p = 1, 3$, and 7, there exist smooth maps $f: M^n \to \mathbf{R}^p$ with only fold singular points such that M^n has odd Euler number. Such an example can be found in [8, Example 3.7], where a smooth map $M^4 \to \mathbf{R}^3$ as above is explicitly constructed. Using a similar method and the fact that $\mathbf{R}P^6$ can be immersed into $\mathbf{R}^7$, for $p = 3, 7$ and $n > p$ with n even, we can construct a smooth map $f: M^n \to \mathbf{R}^p$ with only fold singular points such that M^n has odd Euler number. Note that for $p = 1$, this is obvious, since smooth maps $f: M^n \to \mathbf{R}$ with only fold singular points are nothing but Morse functions.

Remark 2.3. Sakuma conjectures that if M^n is *orientable* and has odd Euler number, then M^n does not admit any smooth map $f: M^n \to \mathbf{R}^p$ $(n \geq p \geq 2)$ with only fold singular points (cf. [9]). Theorem 1.2 shows that this conjecture is true for $p \neq 3, 7$. Note that the examples constructed in Remark 2.2 are not orientable. Thus, for $p = 3, 7$, the problem of Sakuma is still open.

Remark 2.4. In Theorem 1.2, we cannot replace the condition that M^n has odd Euler number with the condition that M^n is not null-cobordant. In fact, there exist manifolds which are not null-cobordant and which admit smooth maps into Euclidean spaces $\mathbf{R}^p$ $(p \neq 1, 3, 7)$ with only fold singular points. For example, the connected sum of two copies of the complex projective plane

$\mathbf{C}P^2 \# \mathbf{C}P^2$ admits a smooth map into $\mathbf{R}^2$ with only fold singular points (see [4]).

We end this paper by a remark concerning the cobordism class of the singular set. In the proof of Theorem 1.2, we used the fact that if M^n has odd Euler number, then the singular set $S(f)$ is not null-cobordant. In fact, the converse of this fact also holds as follows.

Proposition 2.5. *Let $f\colon M^n \to \mathbf{R}^p$ $(n \geq p)$ be a smooth map of a closed n-dimensional manifold M^n into the Euclidean space with only fold singular points. Then the following are equivalent.*

(1) *M^n has even Euler number.*

(2) *$S(f)$ has even Euler number.*

(3) *$S(f)$ is null-cobordant.*

Furthermore, if $S(f)$ is oriented, it is always stably parallelizable. In particular, it is null-cobordant in the oriented category.

Proof. The equivalence of (1) and (2) is easily seen by the fact that the Euler numbers of M^n and $S(f)$ have the same parity [3]. Part (3) obviously implies (2). Thus we have only to show that $S(f)$ is null-cobordant if it has even Euler number. Let $w(S(f)) \in H^*(S(f); \mathbf{Z}_2)$ be the total Stiefel-Whitney class of $S(f)$. Since $S(f)$ admits a codimension-1 immersion into the Euclidean space, we see easily that

$$w(S(f)) = 1 + w_1(S(f)) + w_1(S(f))^2 + \cdots + w_1(S(f))^{p-1},$$

where $w_1(S(f)) \in H^1(S(f); \mathbf{Z}_2)$ is the first Stiefel-Whitney class of $S(f)$. Thus the only Stiefel-Whitney number of $S(f)$ is the one corresponding to $w_1(S(f))^{p-1}$, which is zero by our hypothesis. Hence, $S(f)$ is null-cobordant.

If $S(f)$ is oriented, then the normal bundle of the immersion $f|S(f)$ is trivial and hence $S(f)$ is stably parallelizable. Thus, in this case, $S(f)$ is null-cobordant in the oriented category. □

Remark 2.6. In Theorem 1.1, $S(f)$ is orientable. Thus $S(f)$ is always null-cobordant and M^n has even Euler number. In fact $S(f)$ bounds a canonical p-dimensional orientable manifold, called the Stein factorization of f [7]. Furthermore, in the above proposition, if $p \neq 1, 3, 7$, then (1)–(3) always hold by Theorem 1.2.

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