REPRESENTATION OF A COMPLETELY BOUNDED 
BIMODULE MAP

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ABSTRACT. In this paper, we give a representation for a completely bounded 
$A - B$ bimodule map into $B(H)$, where $A$ and $B$ are unital operator sub-
algebras of $B(H)$. When $A$ and $B$ are $C^*$-subalgebras we give a new proof 
of the Wittstock's theorem by using this representation. We also prove that a 
von Neumann algebra is an injective operator bimodule over its unital operator 
algebras if and only if it is a finitely injective operator bimodule.

1. INTRODUCTION

An operator space is a $L^\infty$-matricially normed space (see [12]). A unital op-
erator algebra is an operator space and is also a unital algebra with completely 
contractive multiplication (see [2]). An operator bimodule over two unital operator 
algebras is an operator space and is also a unital bimodule with completely 
contractive multiplication (see [3]). While there is an extensive literature on the 
representation of completely bounded and related types of linear maps (see [1, 
3,7–10], and others), there has been relatively little done in the way of represen-
tating completely bounded bimodule maps. One notable exception is Smith's 
representation of completely bounded bimodule maps from $K(H)$ into $B(H)$. 
This paper shows in particular that $M_6$ is not an injective operator bimodule 
over a pair of unital operator subalgebras of $M_6$ (see [14]). We are motivated 
by this fact to study the representation of completely bounded bimodule maps 
and the injectivity of $B(H)$ as an operator bimodule.

In §2, we first give a representation for a completely bounded $A - B$ bimodule 
map into $B(H)$ when $A$ and $B$ are $C^*$-subalgebras of $B(H)$. Using this 
representation, we give a new proof of Wittstock's theorem. Later, we generalize 
the representation to the case that $A$ and $B$ are unital operator subalgebras of 
$B(H)$. In §3, we prove that a von Neumann algebra is an injective operator 
bimodule over two unital subalgebras if and only if it is a finitely injective 
operator bimodule.

Throughout this paper, all subspaces, operator subalgebras, operator sub-
bimodules, etc., are closed. We use the term homomorphism for a bimodule

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map when no confusion may result. An embedding is an injective homomorphism. A homeomorphism is a surjective embedding. We do not distinguish between $Y$ an operator subbimodule of $X$ and a completely isometrical embedded copy of $Y$ in $X$. Every vector space is over the complex numbers, and every map is linear.

Suppose $X$ and $Y$ are $A - B$ operator bimodules over unital operator algebras $A$ and $B$. We denote by $\text{Hom}(X, Y)$ the space of all completely bounded homomorphisms from $X$ into $Y$. If $X$ is a subset of a unital $C^*$-algebra, we denote by $C^*(X)$ the unital $C^*$-algebra generated by $X$.

2. Representation of a completely bounded bimodule map

We begin this section with a simple lemma (see [6]).

**Lemma 2.1.** Suppose that $A$ and $B$ are operator algebras with $1_A$ and $1_B$, respectively. Then an operator space $X$ is an $A - B$ operator bimodule if and only if there exists a completely contractive trilinear map $\Phi: A \times X \times B \to X$ that satisfies

$$\Phi(a_1a_2, x, b_1b_2) = \Phi(a_1, \Phi(a_2, x, b_1), b_2)$$

and

$$\Phi(1_A, x, 1_B) = x$$

for all $a_1, a_2 \in A$, $b_1, b_2 \in B$, and $x \in X$. Moreover, the multiplication is determined by $\Phi$ via the equation $\Phi(a, x, b) = axb$ for all $a \in A$, $b \in B$, and $x \in X$.

The following theorem gives us the representations of completely bounded $C^*$-bimodule maps.

**Theorem 2.2.** Suppose that $A$ and $B$ are unital $C^*$-subalgebras of $B(H)$, where $H$ is a Hilbert space. Suppose that $X$ is an $A - B$ operator bimodule. Then every completely bounded $A - B$ bimodule map $\phi$ from $X$ into $B(H)$ has a representation $(V_1, \pi_1, \theta, \pi_2, V_2, K)$, where $\pi_1$ and $\pi_2$ are $*$-representations of $A$ and $B$ on a Hilbert space $K$, $\theta$ is a complete contraction from $X$ into $B(K)$, and $H \overset{V_1}{\to} K \overset{V_2}{\to} H$ are bridging maps such that

$$\phi(x) = V_1\theta(x)V_2;$$

$$\theta(axb) = \pi_1(a)\theta(x)\pi_2(b);$$

$$aV_1 = V_1\pi_1(a), \quad V_2b = \pi_2(b)V_2;$$

$$\|\phi\|_{cb} = \|V_1\|\|V_2\|$$

for all $a \in A$, $x \in X$, and $b \in B$.

**Proof.** Suppose that $(\tilde{\pi}_1, \tilde{\theta}, \tilde{\pi}_2, \tilde{K})$ is a representation of $X$ in Corollary 3.3 of [3], i.e., $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are $*$-representations of $A$ and $B$ on a Hilbert space $\tilde{K}$ and $\tilde{\theta}: X \to B(\tilde{K})$ is a complete isometry such that

$$\tilde{\theta}(axb) = \tilde{\pi}_1(a)\tilde{\theta}(x)\tilde{\pi}_2(b)$$

for all $a \in A$, $x \in X$, and $b \in B$. Applying Lemma 2.1 above, we see that $\tilde{\theta}(X)$ is an $A - B$ operator bimodule with the bimodule multiplication given by $ayb = \tilde{\pi}_1(a)y\tilde{\pi}_2(b)$ for all $a \in A$, $y \in \tilde{\theta}(X)$, and $b \in B$. Moreover, $\tilde{\phi} = \phi \circ \tilde{\theta}^{-1}$ is a completely bounded $A - B$ bimodule map from $\tilde{\theta}(X)$ into
Therefore, there exists a $*$-representation $\pi$ of $B(\tilde{K})$ on some Hilbert space $K$ and bridging maps $H \xrightarrow{\tilde{V}_1} K \xrightarrow{\tilde{V}_2} H$ such that

$$\tilde{\phi}(y) = \tilde{V}_1 \pi(y) \tilde{V}_2$$

for all $y \in \tilde{\theta}(X)$ and $\|\phi\|_{cb} = \|\tilde{V}_1\| \|\tilde{V}_2\|$ (see [7]). Since for each $x \in X$,

$$\phi(x) = \tilde{\phi}(\theta(x)) = \tilde{V}_1 \pi(\tilde{\pi}_1(1)) \pi(\tilde{\theta}(x)) \pi(\tilde{\pi}_2(1)) \tilde{V}_2,$$

we may assume that $\tilde{V}_1 = \tilde{V}_1 \pi(\tilde{\pi}_1(1))$, $\tilde{V}_2 = \pi(\tilde{\pi}_2(1)) \tilde{V}_2$, where $1$ is the unit of $B(H)$.

Let $P: K \to [\pi(\tilde{\theta}(X)) \tilde{V}_2 H]$ be the orthogonal projection onto $[\pi(\tilde{\theta}(X)) \tilde{V}_2 H]$. Then

$$\tilde{\phi} = \tilde{V}_1 \pi \tilde{V}_2 = \tilde{V}_1 P \pi \tilde{V}_2.$$ 

Since $\pi(\tilde{\pi}_1(a)) \pi(\tilde{\theta}(x)) = \pi(\tilde{\theta}(ax))$ for all $a \in A$ and $x \in X$, we have $P \in \pi(\tilde{\pi}_1(A))'$, the commutant of $\pi(\tilde{\pi}_1(A))$. Moreover, since for each $a \in A$ and $x \in X$,

$$\tilde{V}_1 \pi(\tilde{\pi}_1(a)) \pi(\tilde{\theta}(x)) \tilde{V}_2 = \pi \tilde{V}_1 \pi(\tilde{\theta}(x)) \tilde{V}_2,$$

we have

$$\tilde{V}_1 \pi(\tilde{\pi}_1(a)) P = a \tilde{V}_1 P$$

for all $a \in A$. Let $Q: K \to [\pi(\tilde{\theta}(X))^* P \tilde{V}_1^* H]$ be the orthogonal projection onto $[\pi(\tilde{\theta}(X))^* P \tilde{V}_1^* H]$. Since $P \in \pi(\tilde{\pi}_1(A))'$ and

$$\tilde{V}_1 P \pi(\tilde{\theta}(x)) \tilde{V}_2 b = \tilde{V}_1 P \pi(\tilde{\theta}(x)) \pi(\tilde{\pi}_2(b)) \tilde{V}_2$$

for all $x \in X$ and $b \in B$, we have

$$b^* \tilde{V}_2^* Q = \tilde{V}_2^* \pi(\tilde{\pi}_2(b))^* Q$$

by taking adjoints. Thus,

$$Q \tilde{V}_2 b = Q \pi(\tilde{\pi}_2(b)) \tilde{V}_2$$

for all $b \in B$. Since

$$\pi(\tilde{\pi}_2(b))^* \pi(\tilde{\theta}(x))^* P \tilde{V}_1^* h = \pi(\tilde{\theta}(xb))^* P \tilde{V}_1^* h$$

for all $b \in B$, $x \in X$, and $h \in H$, we have $Q \in \pi(\tilde{\pi}_2(B))'$. For any $x \in X$, $h_1$, $h_2 \in H$

$$(\phi(x) h_1, h_2) = (\tilde{V}_1 P \pi(\tilde{\theta}(x)) \tilde{V}_2 h_1, h_2)$$

$$= (\tilde{V}_2 h_1, (\tilde{\theta}(x))^* P \tilde{V}_1^* h_2)$$

$$= (\tilde{V}_2 h_1, Q \pi(\tilde{\theta}(x))^* P \tilde{V}_1^* h_2)$$

$$= (\tilde{V}_1 P \pi(\tilde{\theta}(x)) Q \tilde{V}_2 h_1, h_2).$$

Therefore,

$$\phi(x) = \tilde{\phi}(\theta(x)) = \tilde{V}_1 P \pi(\tilde{\theta}(x)) Q \tilde{V}_2.$$

Now setting $V_1 = \tilde{V}_1 P$, $V_2 = Q \tilde{V}_2$, $\pi_1 = \pi \circ \tilde{\pi}_1$, $\pi_2 = \pi \circ \tilde{\pi}_2$, and $\theta = P(\pi \circ \tilde{\theta}) Q$, we obtain the representation $(V_1, \pi_1, \theta, \pi_2, V_2, K)$ with the properties claimed in the theorem. \qed
Remark 2.1. The representation in Theorem 2.2 depends on the representation of the \(A-B\) operator bimodule \(X\). We will use this to give a new and totally different approach to the proof of Wittstock's theorem (cf. [15, 5]).

Suppose that \(A\) and \(B\) are unital operator algebras, and suppose that \(X\) is an \(A-B\) operator bimodule. Recall that \(X\) is an injective \(A-B\) operator bimodule if for each \(A-B\) operator subbimodule \(Y_1\) of an \(A-B\) operator bimodule \(Y\) and each completely bounded homomorphism \(\phi: Y_1 \to X\) there exists a completely bounded homomorphism \(\tilde{\phi}: Y \to X\) which extends \(\phi\) and has the same cb-norm. In other words, \(X\) is an injective object in the category of \(A-B\) operator bimodules and completely bounded homomorphisms (see [5]).

**Theorem 2.3.** Suppose that \(A\) and \(B\) are unital \(C^*\)-subalgebras of \(B(H)\), where \(H\) is a Hilbert space. Then \(B(H)\) is an injective \(A-B\) operator bimodule.

**Proof.** Suppose that \(X\) is an \(A-B\) operator subbimodule of an \(A-B\) operator bimodule \(Y\). Suppose \(\phi \in \text{Hom}(X, B(H))\). Suppose that \((\tilde{\pi}_1, \tilde{\theta}, \tilde{\pi}_2, K)\) is a representation of \(Y\). Then \((\tilde{\pi}_1, \tilde{\theta}|_X, \tilde{\pi}_2, K)\) is a representation of \(X\). By using the notation in the proof of Theorem 2.2, \(\phi\) has a representation \((V_1, \pi_1, \theta, \pi_2, V_2, K)\) with the properties described there, where \(\hat{\theta} = P(\pi \circ \tilde{\theta}|_X)Q\). Now if we replace \(\hat{\theta}\) by \(\theta = P(\pi \circ \theta)Q\), then it is easy to see that \((V_1, \pi_1, \theta|_X, \pi_2, V_2, K)\) is a representation of \(\phi\) with the properties claimed in Theorem 2.2. Moreover,

\[
\theta(ab) = P\pi(\tilde{\pi}_1(a))\pi(\tilde{\theta}(b))Q = \pi_1(a)\theta(b)\pi_2(b)
\]

for all \(a \in A\), \(x \in X\), and \(b \in B\). Let \(\tilde{\phi}: Y \to B(H)\) be given by the representation \((V_1, \pi_1, \theta, \pi_2, V_2, K)\); i.e., let \(\tilde{\phi} = V_1\theta V_2\). Then \(\tilde{\phi} \in \text{Hom}(Y, B(H))\), extends \(\phi\), and has the same cb-norm \(\|\phi\|_{\text{cb}}\). □

When \(A\) and \(B\) are unital operator algebras, we still have the same form representation for a completely bounded \(A-B\) bimodule map as we do in the case \(A-B\) are unital \(C^*\)-algebras. However, the representation tells less information than it does in the latter case.

**Corollary 2.4.** Suppose that \(A\) and \(B\) are unital operator algebras of \(B(H)\), where \(H\) is Hilbert space. Suppose that \(X\) is an \(A-B\) operator bimodule. Then every completely bounded \(A-B\) bimodule map \(\phi\) from \(X\) into \(B(H)\) has representation \((V_1, \pi_1, \theta, \pi_2, V_2, K)\), where \(\pi_1\) and \(\pi_2\) are \(*\)-representation of \(C^*(A)\) and \(C^*(B)\) on a Hilbert space \(K\), \(\theta\) is a complete contraction from \(X\) into \(B(K)\), and \(H \xrightarrow{V_1} K \xrightarrow{V_2} H\) are bridging maps such that

\[
\phi(x) = V_1\theta(x)V_2; \\
\theta(ab) = \pi_1(a)\theta(x)\pi_2(b); \\
aV_1 = V_1\pi_1(a), \quad V_2b = \pi_2(b)V_2; \\
\|\phi\|_{\text{cb}} = \|V_1\|\|V_2\|
\]

for all \(a \in A\), \(x \in X\), and \(b \in B\).

**Proof.** By a theorem in [6], there exists a completely bounded \(C^*(A) - C^*(B)\) bimodule map \(\tilde{\phi}: \tilde{X} \to B(H)\) such that \(\phi = \tilde{\phi} \circ \alpha\) and \(\|\phi\|_{\text{cb}} = \|\tilde{\phi}\|_{\text{cb}}\), where \(\tilde{X}\)
is a dilation of $X$ which is a $C^*(A) - C^*(B)$ operator bimodule and $\alpha: X \to \tilde{X}$ is a complete contractive $A - B$ bimodule map. Applying Theorem 2.2 to $\tilde{\phi}$ and then restricting to $X$, we get the representation for $\phi$. □

Remark 2.2. It is easy to see that the representation in Corollary 2.4 depends on the dilation $\tilde{X}$ of $X$. We may not use Corollary 2.4 to get an analogous result of Theorem 2.3 when $A$ and $B$ are unital operator algebras. The reason is that when $X$ is an $A - B$ operator sub-bimodule of an $A - B$ operator bimodule $Y$, the dilation $\tilde{X}$ is not necessarily a $C^*(A) - C^*(B)$ operator subbimodule of the dilation $\tilde{Y}$. In fact, $M_6$ is not an $A - B$ operator bimodule for some unital operator subalgebras $A$ and $B$ of $M_6$ (see [14]). The following section will give a sufficient and necessary condition for $B(H)$ to be an injective $A - B$ operator bimodule for unital operator subalgebras $A$ and $B$ of $B(H)$.

3. Injectivity of Operator Bimodules

We say that an $A - B$ operator bimodule is finitely generated if there exists a finite subset $F$ of $X$ such that $X = \sum_{i \in F} A F B$. The concept defined in the following definition seems to be a weaker notion than injectivity.

**Definition 3.1.** An $A-B$ operator bimodule $X$ is called a finitely injective $A-B$ operator bimodule if for any two finitely generated $A-B$ operator bimodules $X_1$ and $X_2$ where $X_1$ is an $A-B$ operator subbimodule of $X_2$ and any $\phi \in \text{Hom}(X_1, X)$ there is a $\phi' \in \text{Hom}(X_2, X)$ which extends $\phi$ and has the same cb-norm. Roughly speaking, $X$ is an injective object in the category of finitely generated $A-B$ operator bimodules and completely bounded homomorphism.

The following theorem shows that injectivity and finite injectivity of operator bimodules are the same for von Neumann algebras. It should provide a useful tool to deal with the injectivity question for operator bimodules.

**Theorem 3.1.** Suppose that $\mathcal{D}$ is a von Neumann algebra. Suppose that $A$ and $B$ are unital operator subalgebras of $\mathcal{D}$. Then $\mathcal{D}$ is an injective $A-B$ operator bimodule if and only if $\mathcal{D}$ is a finitely injective $A-B$ operator bimodule.

**Proof.** It is obvious that injectivity implies finite injectivity. Suppose $\mathcal{D}$ is a finitely injective $A-B$ operator bimodule. We prove $\mathcal{D}$ is an injective $A-B$ operator bimodule. Suppose that $X_1$ is an $A-B$ operator subbimodule of an $A-B$ operator bimodule $X_2$ and $\phi \in \text{Hom}(X_1, \mathcal{D})$. Without loss of generality, we may assume that $\|\phi\|_{cb} = 1$. We claim that for each $x_0 \in X_2 \setminus X_1$ there is a $\phi_{x_0} \in \text{Hom}([AX_1B], \mathcal{D})$ which extends $\phi$ with the same cb-norm.

In fact, we may assume that $\|x_0\| = 1$. Let $\mathcal{F}$ be the family of finite subset of $X_1$. Then $\mathcal{F}$ is a partially ordered set with the usual set-theoretic inclusion partial order. For each $F \in \mathcal{F}$, $\phi_{[A XB]} \in \text{Hom}([AXB], \mathcal{D})$. By the finite injectivity of $\mathcal{D}$, there is an extension $\phi_{x_0,F} \in \text{Hom}([A(F \cup \{x_0\})B], \mathcal{D})$ of $\phi_{[AXB]} \in \text{Hom}([AXB], \mathcal{D})$ such that $\|\phi_{x_0,F} \|_{cb} = \|\phi_{[AXB]}\|_{cb}$. For each $F \in \mathcal{F}$, there is a subset $F_{x_0}$ of $\mathcal{D}$ consisting of all $y \in \mathcal{D}$ such that there is a $\psi \in \text{Hom}([A(F \cup \{x_0\})], \mathcal{D})$ which extends $\phi_{[AXB]}$ with the cb-norm less than or equal to 1 and such that $\psi(x_0) = y$. Then $F_{x_0}$ is a nonempty closed subset of the closed unit ball, $\text{ball}(<\mathcal{D}>)$, of $\mathcal{D}$ which is compact in the weak operator topology. In fact, by the above argument, $F_{x_0} \not= \emptyset$ and $F_{x_0} \subseteq \text{ball}(<\mathcal{D}>)$ because $\|\psi\|_{cb} \leq 1$ and $\|x_0\| = 1$. Suppose that $(y_\lambda)$ is a net in $F_{x_0}$ the converges to
some $y$ in the weak operator topology. Since the ball($\mathcal{D}$) is compact in the weak operator topology, $y \in \text{ball}(\mathcal{D})$. Let $\phi_0 \in \text{Hom}(A(F \cup \{x_0\}B), \mathcal{D})$ be the extension of $\phi|_{AFB}$ such that $\phi_0(x_0) = y$ and $\|\phi_0\| \leq 1$. Since $\phi_0|_{AFB} + Ax_0B$ is totally determined by $y$, the limit $\psi = W\text{-}\lim \phi_0|_{AFB} + Ax_0B$ exists. Since the cb-norm is lower semicontinuous in the weak operator topology, we have $\psi \in \text{Hom}([AFB] + Ax_0B, \mathcal{D})$ and $\|\psi\|_{cb} \leq 1$. Since $[AFB] + Ax_0B$ is norm dense in $[A(F \cup \{x_0\})B]$, $\psi$ may be uniquely continuously extended to $[A(F \cup \{x_0\})B]$. Denoting the extension by $\psi$ also, we have $\psi \in \text{Hom}((A(F \cup \{x_0\})B), \mathcal{D})$, $\|\psi\|_{cb} \leq 1$, and $\psi(x_0) = y$. Therefore, $y \in F_{x_0}$ and $F_{x_0}$ is closed in weak operator topology.

If $\{F_i, x_0, F_i \in \mathcal{F}, 1 \leq i \leq n\}$, $n \in \mathbb{N}$, is a finite subcollection of $\{F_{x_0}, F \in \mathcal{F}\}$, then $\bigcup F_i \in \mathcal{F}$ and $\bigcup F_i \subseteq F_i, x_0$ for all $1 \leq i \leq n$. Therefore, $\{F_{x_0}, F \in \mathcal{F}\}$ has finite intersection property. Since ball($\mathcal{D}$) is compact in the weak operator topology, there is a $y_0 \in \bigcap\{F_{x_0}, F \in \mathcal{F}\}$. Define $\phi_{x_0}: X_1 + Ax_0B \to \mathcal{D}$ in the following way: for each $x \in X_1 + Ax_0B$, there is a $F \in \mathcal{F}$ such that $x \in [A(F \cup \{x_0\})B]$; let $\phi_{F, x_0} \in \text{Hom}([A(F \cup \{x_0\})B], \mathcal{D})$ be such that $\phi_{F, x_0}(x_0) = y_0$ and $\|\phi_{F, x_0}\| \leq 1$; and set $\phi_{x_0}(x) = \phi_{F, x_0}(x)$. That $\phi_{x_0}$ is a well-defined homomorphism that follows from the definition of $y_0$. Moreover, $\|\phi_{x_0}\|_{cb} = \|\phi\|_{cb}$ because $1 = \|\phi\|_{cb} = \sup \|\phi|_{AFB}\|_{cb}$. Since $X_1 + Ax_0B$ is dense in $[X_1 + [Ax_0B]]$, we may continuously extend $\phi_{x_0}$ to $[X_1 + [Ax_0B]]$, obtaining $\tilde{\phi}_{x_0} \in \text{Hom}([X_1 + [Ax_0B]], \mathcal{D})$ which extends $\phi$ with the same cb-norm.

Let $\mathcal{G}$ be the family of pairs $(\phi_Y, Y)$, where $Y$ is an $A - B$ operator sub-bimodule of $X_2$ containing $X_1$ and $\phi_Y \in \text{Hom}(Y, \mathcal{D})$ which extends $\phi$ with the same cb-norm. By the argument just given, $\mathcal{G}$ is a nontrivial family. We give $\mathcal{G}$ the partial order defined by $(\phi_{Y_1}, Y_1) \preceq (\phi_{Y_2}, Y_2)$ if $Y_1$ is an $A - B$ operator sub-bimodule of $Y_2$ and $\phi_{Y_2}|_{Y_1} = \phi_{Y_1}$. By Zorn's lemma, there is a maximal element $(\phi_{Y_0}, Y_0)$. From the initial step, we see that $Y_0 = X_2$. Letting $\tilde{\phi} = \phi_{Y_0}$ yields the desired extension. \hfill \Box

References


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