PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 4, April 1995

ANALYTICITY OF SUBMARKOVIAN SEMIGROUPS

V. A. LISKEVICH AND M. A. PERELMUTER

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let A be a generator of a submarkovian semigroup in $L^2(M, d\mu)$. We investigate the domain of analyticity of $\exp(-tA)$ in $L^p(M, d\mu)$. The same problem for the generator perturbed by potential is considered.

1. INTRODUCTION

Let A be a nonnegative selfadjoint operator in $L^2(M, d\mu)$ such that $\exp(-tA)$ is a bounded semigroup in $L^p(M, d\mu)$ for $\forall p \in [q, q']$, $\frac{1}{q} + \frac{1}{q'} = 1$. Using Stein's interpolation theorem [St], one can show that $\exp(-tA)$ is an analytic semigroup in $L^p(M, d\mu)$, $p \in (q, q')$. Another approach to the proof of the analyticity is based on sectorialness estimates of the following type

$$|\mathrm{Im}\langle A_p f, f|f|^{p-2}\rangle| \leqslant c(p)\, \mathrm{Re}\langle A_p f, f|f|^{p-2}\rangle. \tag{1.1}$$

Inequality (1.1) implies that $\exp(-tA)$ has an analytic continuation as a semigroup in $L^p(M, d\mu)$ in the sector

$$\{z \in \mathbb{C} : |\arg z| < \pi/2 - \arctan c(p)\}$$

(see, e.g., [G]). For second-order elliptic differential operators estimates (1.1) were proved by V. G. Maz′ja and P. E. Sobolevskii [MSo], A. Pazy [P], and other authors. But in mentioned results the constant $c(p)$ depends also on the ellipticity constant and the dimension of underground space. In several particular cases (including Laplace operator) D. Henry [H, §1.6], H. O. Fattorini [F], and D. Bakry [B] proved (1.1) with

$$c(p) = \frac{|p-2|}{2\sqrt{p-1}}, \tag{1.2}$$

i.e., they obtained the constant which depends on p only. Recently N. Okazawa [O] and Yu. A. Semenov [Se] extended this result to the case of

$$A = \sum_{i,j} \partial_i a_{ij} \partial_j \quad \text{in } L^2(\Omega, dx),\ \Omega \subset \mathbb{R}^d.$$

Received by the editors October 12, 1992 and, in revised form, June 29, 1993.
1991 *Mathematics Subject Classification.* Primary 47D07, 47D03, 47A55.
Key words and phrases. Submarkovian semigroups, analytic semigroups.
The first author is a recipient of a Dov Biegun Postdoctoral Fellowship.

D. Bakry and N. Okazawa have also pointed out that the analyticity sector of the corresponding semigroup, which can be obtained by using (1.1), is wider than that obtained by interpolation method.

The aim of this paper is to prove (1.1) with $c(p)$ from (1.2) for any submarkovian generator. In many respects this paper is complementary to [LPSe, LSe1]; we use the technique from these papers and extend some results. Here we do not give examples, but each example from [LPSe] may be considered in the spirit of this article. Note also that the operators considered in [F, H, MSo, P] are particular cases of submarkovian generators.

2. Some preliminaries

In this section we give some notation, definitions, and lemmas which will be needed later on.

Let (M, μ) be a σ-finite measure space. We use the following notation: $L^p \equiv L^p(M, \mu)$; $\|\cdot\|_p$ is the norm in L^p;

$$\langle f, g\rangle \equiv \int_M f(x)\overline{g(x)}\, d\mu(x) .$$

Definition 2.1. We say that A is a generator of a submarkovian semigroup (submarkovian generator) if the following conditions are satisfied:

(i) A is a nonnegative selfadjoint operator in L^2.
(ii) $\|e^{-tA}f\|_\infty \le \|f\|_\infty$, $\forall\, f \in L^1 \cap L^\infty$.
(iii) $0 \le f \in L^2 \Longrightarrow e^{-tA}f \ge 0$ almost everywhere.

Let A be a selfadjoint operator in L^2. Suppose that

$$\|e^{-tA}f\|_p \leqslant N\, e^{\omega t}\|f\|_p, \quad \forall f \in L^1 \cap L^\infty,\ \forall p \in [p_0, p_0'],\ p_0 \in [1, 2),$$

for some $N, \omega < \infty$ (primes always denote the dual exponent, i.e., $\frac{1}{p}+\frac{1}{p'} = 1$). Assume that $T_p^t = \exp(-tA_p)$ is the closure of $\exp(-tA) \restriction [L^2 \cap L^p]$ in the space L^p, $p \in [1, \infty)$, and $T_\infty^t = \exp(-tA_1)^*$ (adjoint operator). (Thus $A_2 \equiv A$.)

Lemma 2.2. *For any $1 < p < \infty$ and $x, y \in \mathbb{C}$ the following inequality holds true*:

$$(2.1)\quad |\mathrm{Im}(x-y)(\bar{x}|x|^{p-2} - \bar{y}|y|^{p-2})| \leqslant \frac{|p-2|}{2\sqrt{p-1}}\mathrm{Re}(x-y)(\bar{x}|x|^{p-2} - \bar{y}|y|^{p-2}).$$

Proof. Let $z = (x-y)(\bar{x}|x|^{p-2} - \bar{y}|y|^{p-2})$; then

$$\mathrm{Im}\, z = (|x|^{p-2} - |y|^{p-2})\mathrm{Im}(x\bar{y}) = (|x|^{p-2} - |y|^{p-2})|x||y| \sin\omega,$$

$$\begin{aligned}\mathrm{Re}\, z &= |x|^p + |y|^p - (|x|^{p-2} + |y|^{p-2})\mathrm{Re}\,(x\bar{y}) \\ &= |x|^p + |y|^p - |x||y|(|x|^{p-2} + |y|^{p-2})\cos\omega,\end{aligned}$$

where $\omega = \arg(x\bar{y})$.

It is clear that $\mathrm{Re}\, z \geqslant 0$. Let $s = |x|$ and $t = |y|$; then we can rewrite (2.1) in the form

$$\sup_{s, t \in [0, \infty)} \sup_{\omega \in (-\pi, \pi]} F(s, t, \omega) \leqslant \frac{(p-2)^2}{4(p-1)},$$

where

$$F(s, t, \omega) = \frac{(s^{p-2} - t^{p-2})^2 s^2 t^2 (1-\cos^2\omega)}{(s^p + t^p - st(s^{p-2}+t^{p-2})\cos\omega)^2}.$$

Optimizing $F(s, t, \omega)$ with respect to $\cos\omega$, i.e., setting

$$\cos\omega = \frac{st(s^{p-2} + t^{p-2})}{s^p + t^p},$$

we obtain

$$F(s, t, \omega) \leqslant \frac{(s^{p-2} - t^{p-2})^2 s^2 t^2}{(s^2 - t^2)(s^{2p-2} - t^{2p-2})}. \tag{2.2}$$

Now using Schwarz's inequality we have

$$\begin{aligned}(s^{p-2} - t^{p-2})^2 &= (p-2)^2\Big(\int_t^s z^{p-3}\,dz\Big)^2 = (p-2)^2\Big(\int_t^s z^{p-\frac{3}{2}} z^{-\frac{3}{2}}\,dz\Big)^2 \\ &\le (p-2)^2 \int_t^s z^{2p-3}\,dz \int_t^s z^{-3}\,dz \\ &= \frac{(p-2)^2}{4(p-1)}(s^{2p-2} - t^{2p-2})(s^{-2} - t^{-2}).\end{aligned}$$

Substituting this inequality into (2.2) yields

$$F(s, t, \omega) \leqslant \frac{(p-2)^2}{4(p-1)}. \quad \square$$

Lemma 2.3. *For any $1 < p < \infty$ and $x, y \in \mathbb{C}$ the following inequality holds true*:

$$\begin{aligned}4\frac{p-1}{p^2}\big|x|x|^{\frac{p}{2}-1} - y|y|^{\frac{p}{2}-1}\big|^2 &\le \operatorname{Re}(x-y)(\bar{x}|x|^{p-2} - \bar{y}|y|^{p-2}) \\ &\le a(p)\big|x|x|^{\frac{p}{2}-1} - y|y|^{\frac{p}{2}-1}\big|^2,\end{aligned} \tag{2.3}$$

where

$$a(p) = \sup_{u\in[0,1]} \frac{(u^{1/p} + 1)(u^{1/p'} + 1)}{(u^{\frac{1}{2}} + 1)^2}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. Let $A = \operatorname{Re}\,(x-y)(\bar{x}|x|^{p-2} - \bar{y}|y|^{p-2})$, $B = |x|x|^{\frac{p}{2}-1} - y|y|^{\frac{p}{2}-1}|^2$. Then

$$A = |x|^p + |y|^p - (|x|^{p-2} + |y|^{p-2})\operatorname{Re}(x\bar{y}),$$

$$B = |x|^p + |y|^p - 2|x|^{\frac{p}{2}-1}|y|^{\frac{p}{2}-1}\operatorname{Re}(x\bar{y}).$$

Let us denote $s = |x|$, $t = |y|$, $\arg(x\bar{y}) = \omega$, and $\cos\omega \equiv \alpha \in [-1, 1]$. Then

$$A = s^p + t^p - st(s^{p-2} + t^{p-2})\alpha, \qquad B = s^p + t^p - 2s^{\frac{p}{2}}t^{\frac{p}{2}}\alpha.$$

Inequality (2.3) takes the form

$$4\frac{p-1}{p^2} \leqslant \frac{A}{B} \leqslant a(p).$$

Let

$$G(s, t, \alpha) \equiv \frac{A}{B} = \frac{s^p + t^p - st(s^{p-2} + t^{p-2})\alpha}{s^p + t^p - 2s^{\frac{p}{2}}t^{\frac{p}{2}}\alpha}.$$

Simple computations show that

$$\frac{\partial G(s, t, \alpha)}{\partial \alpha} \leqslant 0.$$

Thus using inclusion $\alpha \in [-1, 1]$, we have

$$G(s, t, 1) \leqslant \frac{A}{B} \leqslant G(s, t, -1).$$

By Schwarz's inequality we get

$$\begin{aligned} G(s, t, 1) &= (s-t)(s^{p-1} - t^{p-1})(s^{\frac{p}{2}} - t^{\frac{p}{2}})^{-2} \\ &= 4\frac{p-1}{p^2} \frac{\int_t^s dz \int_t^s z^{p-2} dz}{(\int_t^s z^{\frac{p}{2}-1} dz)^2} \geqslant 4\frac{p-1}{p^2}. \end{aligned}$$

It remains only to note that

$$\begin{aligned} G(s, t, -1) &= (s^p + t^p + st(s^{p-2} + t^{p-2}))(s^{\frac{p}{2}} + t^{\frac{p}{2}})^{-2} \\ &\leq \sup_{s, t \in [0, \infty)} (s^p + t^p + st(s^{p-2} + t^{p-2}))(s^{\frac{p}{2}} + t^{\frac{p}{2}})^{-2} \\ &= \sup_{u \in [0, 1]} \frac{(u^{1/p} + 1)(u^{1/p'} + 1)}{(u^{\frac{1}{2}} + 1)^2}. \quad \square \end{aligned}$$

Remarks. 1. If x, y are real and of the same sign, then $a(p)$ may be changed by 1. Indeed, in this case $\alpha = 1$ and the inequality $A \leqslant B$ is a consequence of the Young inequality.

2. The function $a(p)$ has the following properties: $a(1) = 2$; $a(2) = 1$; $a(p) = a(p')$, $1 \leq a(p) \leq 2$ for $1 < p < \infty$; $a(p)$ decreases on $(1, 2)$ and increases on $(2, \infty)$.

For real x, y the left-hand side of (2.3) is a particular case of the more general inequality of the following type:

$$(x - y)(\phi(x) - \phi(y)) \geqslant \mathrm{const}(\psi(x) - \psi(y))^2.$$

A variant of this inequality was proved in [LSe1]. Here we give a different version:

Lemma 2.4. *Let $\phi : [a, \infty) \longmapsto \mathbb{R}^1$ be a nondecreasing function and function ψ be such that $(\psi')^2 \leqslant \phi'$; then for any $x, y \in [a, \infty)$*

$$(x - y)(\phi(x) - \phi(y)) \geqslant (\psi(x) - \psi(y))^2.$$

Proof. Using Schwarz's inequality we have

$$\begin{aligned} (\psi(x) - \psi(y))^2 &= \left(\int_y^x \psi'(z) dz\right)^2 \leqslant \left(\int_y^x dz\right) \cdot \left(\int_y^x \psi'(z)^2 dz\right) \\ &\leq \left(\int_y^x dz\right) \cdot \left(\int_y^x \phi'(z) dz\right) = (x - y)(\phi(x) - \phi(y)). \quad \square \end{aligned}$$

Remarks. 1. If $\phi(x) = x|x|^{p-2}$ and $\psi(x) = \frac{2\sqrt{p-1}}{p} x|x|^{\frac{p}{2}-1}$, we obtain the left-hand side of (2.3).

2. Another example: if $\phi(x) = \ln(x)$ and $\psi(x) = 2\sqrt{x}$, $a = 0$ leads to the inequality $(x-y)(\ln x - \ln y) \geq 4(\sqrt{x} - \sqrt{y})$, which together with (2.3) is useful for abstraction of Moser's proof of Harnack inequality for second-order elliptic partial differential equations.

3. Analyticity of semigroups

In this section we give some consequences of the elementary inequalities from the preceding section.

Theorem 3.1. *Let A be a submarkovian generator. If $f \in \mathscr{D}(A_p)$ for some $p \in (1, +\infty)$, then the following inequality holds true*:

$$|\mathrm{Im}\,\langle A_p f, f|f|^{p-2}\rangle| \leqslant \frac{|p-2|}{2\sqrt{p-1}} \mathrm{Re}\langle A_p f, f|f|^{p-2}\rangle. \tag{3.1}$$

Proof. Let $T^t =: T^t_\infty$. For any measurable set $G \subset M$ we set $P(t, \cdot, G) = T^t \chi_G$, where χ is the characteristic function of G. $P(t, \cdot, G)$ is a finitely additive set function and $P(t, \cdot, M) \leq 1$.

For any simple function $f = \sum_{i=1}^k c_i \chi_{G_i}$, where $\{G_i\}$ are disjoint sets of finite measure, $c_i \in \mathbb{C}$, let us define

$$\int_M P(t, \cdot, dy) f(y) = T^t f = \sum_{i=1}^k c_i T^t \chi_{G_i}.$$

Then

$$T^t f|f|^{p-2} = \sum_{i=1}^k c_i |c_i|^{p-2} T^t \chi_{G_i}.$$

Let $\epsilon_t(u, v) = \frac{1}{t}\langle(1 - T^t)u, v\rangle$. Because of selfadjointness of A and, consequently, of symmetry in (x, y) of the finite additive product-measure $d\mu_t(x, y) = P(t, x, dy)\, d\mu(x)$, the following equalities are valid:

$$\begin{aligned}
\langle T^t f, f|f|^{p-2}\rangle &= \langle f, T^t f|f|^{p-2}\rangle \\
&= \frac{1}{2}\int d\mu(x) \int P(t, x, dy)(f(x)\overline{f(y)}|f(y)|^{p-2} \\
&\qquad\qquad + f(y)\overline{f(x)}|f(x)|^{p-2}); \\
\langle (T^t\chi_E) f, f|f|^{p-2}\rangle &= \langle \chi_E, T^t|f|^p\rangle \\
&= \frac{1}{2}\int d\mu(x) \int P(t, x, dy)(|f(x)|^p + |f(y)|^p),
\end{aligned}$$

where $E = \bigcup_{i=1}^k G_i$. Hence we obtain

$$\begin{aligned}
\epsilon_t(f, f|f|^{p-2}) = &\frac{1}{2t}\int d\mu(x) \int P(t, x, dy)(f(x) - f(y)) \\
&\qquad \times (\overline{f(x)}|f(x)|^{p-2} - \overline{f(y)}|f(y)|^{p-2}) \\
&+ \frac{1}{t}\langle(1 - T^t\chi_E), |f|^p\rangle .
\end{aligned}$$

Using Lemma 2.2, we have

$$|\mathrm{Im}\,\epsilon_t(f, f|f|^{p-2})| \leqslant \frac{|p-2|}{2\sqrt{p-1}} \mathrm{Re}\,\epsilon_t(f, f|f|^{p-2}). \tag{3.2}$$

Since the set of simple functions is dense in L^q $(1 \leqslant q < \infty)$, $f|f|^{p-2} \in L^{p'}$, thus (3.2) holds true for all $f \in L^p$. If $f \in \mathscr{D}(A_p)$, letting $t \to 0$ in (3.2) yields (3.1). □

It is well known (see, e.g., [G, Theorem 1.5.9]) that (3.1) implies

Corollary 3.2. *Let A be a submarkovian generator; then A_p $(1 < p < \infty)$ generates a semigroup which is analytic in the sector*

$$E_p = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}} \right\}$$

and $\|\exp(-zA_p)\|_{p,p} \leqslant 1$, $\forall z \in E_p$.

Remarks. 1. Stein's interpolation theorem implies that A_p generates a semigroup on L^p which is analytic in the sector

$$S_p = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right)\right\}$$

(see [St]), but simple calculations show that $S_p \subset E_p$; thus Corollary 3.2 improves Stein's result in the submarkovian case.

2. Corollary 3.2 does not give any information for the case $p = 1$. Nevertheless the analyticity of the semigroup generated by second-order elliptic operators holds true for quite general conditions on the coefficients [KoSe].

Theorem 3.3. *Let A be a submarkovian generator. If $f \in \mathscr{D}(A_p)$ for some $p \in (1, +\infty)$ then $f|f|^{\frac{p}{2}-1} \equiv g_p \in \mathscr{D}(A^{\frac{1}{2}})$ and the following inequality holds true:*

$$4\frac{p-1}{p^2}\|A^{\frac{1}{2}} g_p\|_2^2 \le \operatorname{Re}\langle A_p f, f|f|^{p-2}\rangle \le a(p)\|A^{\frac{1}{2}} g_p\|_2^2 \tag{3.3}$$

where $a(p)$ is from (2.3) *and if $f \ge 0$, then $a(p) = 1$.*

The proof is similar to that of Theorem 3.1 and is based on Lemma 2.3.

Remarks. 1. The inequalities (3.3) for $f \ge 0$ are due to [S, V, CKS]. In the case $\operatorname{Im} f = 0$ the generalization for the submarkovian generators acting in L^p was proved in [LSe1].

2. Generally, the constant $a(p)$ depends on operator A. For example, if A is an operator of multiplication by nonnegative measurable function from L^1, then $a(p)$ may be replaced by 1.

4. Perturbations

In this section we investigate the perturbation of a submarkovian generator A by a potential, e.g., by the operator of multiplication by a measurable function.

Let V be a real-valued measurable function such that $V = V^+ - V^-$, $V^\pm \geqslant 0$.

We need the following

Definition 4.1. Let A be a submarkovian generator. Then we write $V^- \in PK_\beta(A)$ if $\mathscr{D}((V^-)^{\frac{1}{2}}) \supset \mathscr{D}(A^{\frac{1}{2}})$ and

$$\|(V^-)^{\frac{1}{2}}\varphi\|_2^2 \le \beta\|A^{\frac{1}{2}}\varphi\|_2^2 + C(\beta)\|\varphi\|_2^2, \quad \forall \varphi \in \mathscr{D}(A^{\frac{1}{2}})$$

for some $\beta \in [0, 1)$, $C(\beta) \ge 0$.

If $\mathscr{D}((V^+)^{\frac{1}{2}}) \cap \mathscr{D}(A^{\frac{1}{2}})$ is dense in L^2 and $V^- \in PK_\beta(A)$, then the form-sum $H \equiv A + V$ is well defined (see [K, Chapter 6]).

As it was shown in [LSe1, LPSe], if $V^- \in PK_\beta(A)$, $p \in [t(\beta), t'(\beta)]$, where $t(\beta) = \frac{2}{1+\sqrt{1-\beta}}$ and $t'(\beta) = \frac{2}{1-\sqrt{1-\beta}}$, then

$$\| \exp(-t(A+V+C(\beta))f\|_p \leqslant \|f\|_p, \quad \forall f \in L^2 \cap L^p.$$

Thus we can define the operator $H_p \equiv (A+V)_p$, where H_p is the generator of the semigroup which is the closure of $\exp(-tH) \restriction [L^2 \cap L^p]$.

Theorem 4.2. *Let $\beta \in [0, 1)$, $t(\beta) = \frac{2}{1+\sqrt{1-\beta}}$ and $t'(\beta) = \frac{2}{1-\sqrt{1-\beta}}$, $p \in (t(\beta), t'(\beta))$. Suppose that $\mathscr{D}((V^+)^{\frac{1}{2}}) \cap \mathscr{D}(A^{\frac{1}{2}})$ is dense in L^2 and $V^- \in PK_\beta(A)$. Let $C_p = (A+V)_p + C(\beta)$; then*

$$|\mathrm{Im}\langle C_p f, f|f|^{p-2}\rangle| \leqslant s\, \mathrm{Re}\langle C_p f, f|f|^{p-2}\rangle, \quad \forall f \in \mathscr{D}(C_p), \tag{4.1}$$

where

$$s = \frac{|p-2|}{2\sqrt{p-1}} \frac{4(p-1)}{4(p-1) - \beta p^2}.$$

Hence e^{-tC_p} is the semigroup which is analytic in the sector $S = \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2} - \arctan s\}$ and $\|e^{-zC_p}\|_{p,p} \leqslant 1$, $\forall z \in S$.

Proof. Let $V_n^+ = \max\{V^+, n\}$, $V_m^- = \max\{V^-, m\}$. For any $f \in \mathscr{D}(C_p)$ we have $f = (C_p+1)^{-1}u$, where $u \in L^p$. Let $f_{n,m} = (1 + C_p^{(n,m)})^{-1}u$, where $C_p^{(n,m)} = A_p + V_n^+ - V_n^- + C(\beta)$. Using the general approximation theorems (see [K, Chapter VI]) it is easy to check that $f_{n,m} \longrightarrow f$ in L^2 as $n, m \to \infty$ and simple arguments based on Hölder's inequality show that the same is true in L^p [LPSe].

Then $C_p^{(n,m)}(C_p^{(n,m)} + 1)^{-1}u \to C_p f$ (strongly in L^p) and $f_{n,m}|f_{n,m}|^{p-2} \to f|f|^{p-2}$ (strongly in $L^{p'}$.) Thus

$$\langle C_p f, f|f|^{p-2}\rangle = \lim_{n,m\to\infty} \langle C_p^{(n,m)} f_{n,m}, f_{n,m}|f_{n,m}|^{p-2}\rangle.$$

So in the proof of (4.1) we may assume that V is bounded and in all subsequent calculations we may assume also that all operators have the same domain. By Theorem 3.1

$$|\mathrm{Im}\langle C_p f, f|f|^{p-2}\rangle| \leqslant \frac{|p-2|}{2\sqrt{p-1}} \mathrm{Re}\langle A_p f, f|f|^{p-2}\rangle \quad \forall f \in \mathscr{D}(A_p).$$

Using the definition of $PK_\beta(A)$ and Theorem 3.3 we obtain $\forall f \in \mathscr{D}(A_p)$

$$\begin{aligned} &\mathrm{Re}\langle (A_p + V^+ - V^- + C(\beta))f, f|f|^{p-2}\rangle \\ &\geqslant \left(1 - \beta\frac{p^2}{4(p-1)}\right) \mathrm{Re}\langle A_p f, f|f|^{p-2}\rangle. \quad \square \end{aligned}$$

Remark. More general classes of perturbations of submarkovian generators will be considered in [LSe2].

Acknowledgement

This work was completed while the second author was visiting the Department of Theoretical Mathematics at the Weizmann Institute. It is a pleasure to thank the Department for its support and hospitality.

REFERENCES

[B] D. Bakry, *Sur l'interpolation complexe des semigroupes de diffusion*, Seminaire de Probabilites, XXIII, Lecture Notes in Math., vol. 1372, Springer-Verlag, Berlin and New York, 1989, pp. 1–20.

[CKuS] E. A. Carlen, S. Kusuoka, and D. W. Stroock, *Upper bounds for symmetric Markov transition functions.*, Ann. Inst. Henri Poincaré **23** (1987), 245–287.

[F] H. O. Fattorini, *On the angle of dissipativity of ordinary and partial differential operators*, Functional Analysis, Holomorphy and Approximation Theory. II (G. I. Zapata, ed.), North-Holland Math. Stud., vol. 86, North-Holland, Amsterdam and New York, 1984, pp. 85–111.

[G] J. Goldstein, *Semigroups of linear operators and applications*, Oxford Univ. Press, New York, 1985.

[H] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin and New York, 1981.

[K] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin and Heidelberg, 1966.

[KoSe] V. F. Kovalenko and Yu. A. Semenov, *On the L^1-theory of parabolic semigroups*, Ukrain. Math. J. **41** (1989), 1375–1389.

[LPSe] V. A. Liskevich, M. A. Perelmuter, and Yu. A. Semenov, *Form-bounded perturbations of generators of submarkovian semigroups*, submitted.

[LSe1] V. A. Liskevich and Yu. A. Semenov, *Some inequalities for submarkovian generators and their applications to the perturbation theory*, Proc. Amer. Math. Soc. **119** (1993), 1171–1177.

[LSe2] ———, *Some problem on Markov semigroup*, in preparation.

[MSo] V. G. Maz'ja and P. E. Sobolevskii, *On generators of operators semigroups*, Uspekhi Mat. Nauk **17** (1962), 151–154. (Russian)

[O] N. Okazawa, *Sectorialness of second order elliptic operators in divergence form*, Proc. Amer. Math. Soc. **113** (1991), 701–706.

[P] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.

[Se] Yu. A. Semenov, private communication.

[St] E. M. Stein, *Topics in harmonic analysis related to Littlewood-Paley theory*, Princeton Univ. Press, Princeton, NJ, 1970.

[S] D. W. Stroock, *An introduction to the theory of large deviations*, Springer-Verlag, New York, 1984.

[V] N. Th. Varopoulos, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal. **63** (1985), 240–260.

DEPARTMENT OF THEORETICAL MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL
E-mail address: `vitaliy@wisdom.weizmann.ac.il`

RESEARCH INSTITUTE OF AUTOMATIZED SYSTEMS FOR PLANNING & CONTROL IN CONSTRUCTION, M.KRIVONOSA 2A, KIEV, 252180, UKRAINE