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INTEGRAL CLOSURES OF NOETHERIAN INTEGRAL DOMAINS AS INTERSECTIONS

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ABSTRACT. Three equivalent formulations of the property that the integral closure $\overline{A}$ of a noetherian domain A can be written as $\bigcap \overline{A}_p$ at all height-one primes p, are given in terms of PDE, $A^{(1)}$, and bad minimal primes in completions. Examples with these properties include excellent domains and domains with a canonical module. Writing $\overline{A}$ as an intersection of DVR's is also addressed.

INTRODUCTION

If an affine domain A (i.e., finitely generated as an algebra over a field) is such that A_p is a DVR for all p in the set G of height-one primes of A, then its integral closure $\overline{A}$ can be written as $A^{(1)} := \bigcap_{p\in G} A_p$. For which noetherian domains does $\overline{A} = A^{(1)}$ (in which case A_p must be a DVR for all $p \in G$)? In this note we investigate a description of when $\overline{A} = \bigcap_{p\in G} \overline{A}_p$ in terms of PDE, $A^{(1)}$, and bad formal minimal primes, that help answer this question. Here, $\overline{A}_p$ means $(A_p)^-$ and equals $(\overline{A})_p$, the localization of $\overline{A}$ at the multiplicatively closed subset $A - p$ of A [S, Corollary 13.27]. Examples include excellent domains and domains with (locally) a canonical module such as a homomorphic image of a Gorenstein ring.

MAIN RESULTS

The following lemma holds for any domains R and S, and in torsion theoretic language says simply that $Q_Y(S) = Q_{Y'}(S)$.

Lemma. *Let $R \subseteq S$ be domains, and let $Y \subseteq \operatorname{Spec} R$ be nonempty and generically closed (i.e., $p' \subseteq p \in Y \Rightarrow p' \in Y$). Set $Y' = \{q \in \operatorname{Spec} S | q \cap R \in Y\}$ (thus, Y' is also generically closed and nonempty). Then $\bigcap_{p\in Y} S_p = \bigcap_{q\in Y'} S_q$.*

Proof. It is clear the "$\subseteq$" holds since, for each $q \in Y'$, we have $S_p \subseteq S_q$ where $p = q \cap R$.

Conversely, if $x \in \cap S_q$, let $J = (S :_S x)$. For every $q \in Y'$, $J \nsubseteq q$ since $x \in S_q$. Then $I = J \cap R$ is not contained in any $p \in Y$, for if I is disjoint from the multiplicatively closed subset $R - p \subseteq S$, then so is J, and J can

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be expanded to a prime q' (not in Y' by a preceding remark) disjoint from $R-p$, i.e., $q' \cap R \subseteq p \in Y$. But then $q' \cap R \in Y$, so $q' \in Y'$, a contradiction. Therefore, for each $p \in Y$ there is an element $a \in I - p$ with $ax \in S$, proving $x \in S_p$.

Theorem. *Let A be a noetherian integral domain, $\overline{A}$ its integral closure, and set $G = \{p \in \operatorname{Spec} A \mid \operatorname{ht} p \leq 1\}$. The following conditions are equivalent:*

(1) $\overline{A} = \bigcap_{p \in G} \overline{A}_p$.

(2) *Height-one primes of $\overline{A}$ contract to height-one primes of A.*

(3) $A \subseteq \bigcap_{p \in G} A_p$ *is an integral extension.*

(4) *For all maximal ideals m of A, $B := A_m$, if $q \subset q' \in \operatorname{Spec} \widehat{B}$ with q minimal and $\operatorname{ht}(q' \cap B) \geq 2$, then there is a prime strictly between q and q'.*

Proof. (1) $\Leftrightarrow$ (2). For integral extensions, Lying-Over holds and the height of a contracted prime cannot decrease (by INC), so all the primes of $\overline{A}$ lying over a height-one prime of A are height one [M, Exercise 9.8 and Theorem 9.3(i)]. Set $H_1 = \{q \in \operatorname{Spec} \overline{A} \mid \operatorname{ht}(q) \leq 1\}$ and $H = \{q \in H_1 \mid \operatorname{ht}(q \cap A) \leq 1\}$. Statement (2) is then $H = H_1$. We now use the fact that $\overline{A}$ is Krull [F, Theorem 4.3]. By the lemma, we have the equation

$$\overline{A} = \bigcap_{q \in H_1} (\overline{A})_q \subseteq \bigcap_{q \in H} (\overline{A})_q = \bigcap_{p \in G} (\overline{A})_p = \bigcap_{p \in G} \overline{A}_p,$$

with the inclusion an equality if and only if $H = H_1$ [M, Theorem 12.3]. Thus, (1) holds if and only if $H = H_1$ which is (2).

(2) $\Leftrightarrow$ (3). We use the notation $Q_U(A) := \bigcap_{p \in U} A_p$ and assume without loss of generality that A is local (observe: $Q_G(A)/A = \bigcup (A :_K I)/A$ the union taken over all ideals I of A not contained in any prime of G; this equality localizes at any prime p since each I is finitely generated). If $x \in Q_G(A)$, $I = (A :_A x)$, and U is the open set $D(I) \supseteq G$, then $x \in Q_U(A) \subseteq Q_G(A)$. Thus $Q_G(A) = \bigcup Q_U(A)$, the union taken over all open $U \supseteq G$. Now [NII, Theorem 2.6.1] says $Q_U(A)$ is integral if and only if for each $p \notin U$, $Q_{Y(p)}(A_p)$ is integral over A_p, where $Y(p)$ is the punctured spectrum of A_p.

On the other hand, (2) is equivalent to saying if $\dim A_p \geq 2$, there are no height-one maximal ideals of $\overline{A}_p$.

Thus (with a change of notation), by the above two reductions, we need to show whenever $\dim A \geq 2$ that $Q_Y(A)$ is integral over A where Y is the punctured spectrum of A, if and only if there are no height-one maximals of $\overline{A}$. But this is [NI, Corollary 1.7].

(3) $\Leftrightarrow$ (4). Again, without loss of generality, A is a local domain and for any open set U containing G we want to show that $Q_U(A)$ is integral over A if and only if $\dim \widehat{A}_{q'}/q > 1$ for all primes $q \subsetneq q'$ of $\widehat{A}$ with q minimal and $q' \cap A \notin U$. This is just [NII, Proposition 2.7.1].

Remark 1. For a simpler proof of [NII, Proposition 2.7.1] used above in the discussion of the integrality of $Q_U(A)$, start with Brodmann's consideration of the A-module finiteness of $Q_U(A)$ in [B, Corollary 3.7]. Then the integrality statement follows from the clever observation of Ferrand and Raynaud of reducing to the reduced case [FR, Proposition 1.1]. Also see [Mc, Chapter X].

Remark 2. PDE. Let us say for domains $A \subseteq B$ that the extension *satisfies PDE* if each height-one prime of B contracts to a prime of height ≤ 1 in A (e.g., as in condition (2) of theorem). In fact, when A is noetherian and condition (2) holds, every integral extension B of A that is also Krull satisfies PDE (as do all extensions between B and A, e.g., as in (3) of the theorem). This follows easily from the known result that if $A \subseteq B$ is an integral extension with A and B *both* Krull, then PDE holds [F, Proposition 6.4(b)]. For example, if a domain $B \supseteq A$ is module finite (hence integral) over a noetherian domain A, then B is noetherian and $\overline{B}$ is Krull. If the Theorem applies to A, then PDE holds for $A \subseteq \overline{B}$, hence for $B \subseteq \overline{B}$. This proves:

Corollary 1. *If $A \subseteq B$ are domains, A noetherian and B module finite over A, and A satisfies one of the equivalent conditions of the Theorem, then so does B.*

Examples. Locally, when the noetherian domain A has a canonical module, then the extension in (3) of the Theorem is module finite [A, Theorem 3.2], hence integral (cf. [B, Proposition 5.2]). Examples include complete noetherian local domains and domains finitely generated as an algebra over a field. More generally, any domain that is the homomorphic image of a Gorenstein ring has a canonical module.

If A is an excellent (local) domain, then it is universally catenary, hence $\widehat{A}$ is equidimensional (and catenary, of course) and thus (4) holds [M, Theorem 31.7].

In the case that one of the above examples also has the property that A_p is a DVR for each height-one prime p, then $\overline{A} = \bigcap_{p \in G} A_p$.

For a converse we have

Corollary 2. *Let A be any noetherian domain and H any set of height-one primes such that A_p is a DVR for each $p \in H$. If $\overline{A} = \bigcap_{p \in H} A_p$, then conditions (1)–(4) hold, A_p is a DVR for every height-one prime p of A, and H is precisely the set of all height-one primes of A.*

Proof. Since for each $p \in H$, $(\overline{A})_p = (A_p)^- = A_p$, there is a unique prime $q \in \operatorname{Spec} \overline{A}$ lying over p, and it is of height one since $(\overline{A})_q = A_p$. Now $\overline{A}$ is Krull, so the primes q so obtained must in fact be all the height-ones of $\overline{A}$. Thus condition (2) holds. LO and INC give the last statements in the corollary since height cannot decrease upon contraction.

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