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ON A MULTIPLICITY FORMULA OF WEIGHTS OF REPRESENTATIONS OF $SO^*(2n)$ AND RECIPROCITY THEOREMS FOR SYMPLECTIC GROUPS

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ABSTRACT. A formula relating the multiplicity of a weight of a holomorphic discrete series of the group $SO^*(2n)$ to the frequency of occurrence of an irreducible holomorphically induced representation of the group $Sp(2n, \mathbb{C})$ in an n-fold tensor product of irreducible symmetric representations of $Sp(2n, \mathbb{C})$ is given. Reciprocity theorems relating holomorphic discrete series of $SO^*(2n)$ (resp. $Sp(2n, \mathbb{R})$) to holomorphically induced representations of $Sp(2k, \mathbb{C})$ (resp. $SO(N, \mathbb{C})$) are also derived.

1. INTRODUCTION

Let $GL(n, \mathbb{C})$ denote the complex general linear group of order n, let $SO(n, \mathbb{C})$ and $U(n)$ be its complex special orthogonal and unitary subgroups, respectively. The groups $Sp(2k, \mathbb{C})$ and $SO^*(2k)$ can be realized, respectively, as the subgroup of $GL(2k, \mathbb{C})$ which leaves invariant the symplectic bilinear form $\sum_{i=1,\dots,k}(-Z_iW_{2k-i+1} + W_iZ_{2k-i+1})$ and the subgroup of $SO(2k, \mathbb{C})$ which leaves invariant the skew Hermitian form $\sum_{i=1,\dots,k}(-Z_i\bar{W}_{k+i} + Z_{k+i}\bar{W}_i)$ for all $(Z_1, \dots, Z_{2k})$ and $(W_1, \dots, W_{2k})$ in $\mathbb{C}^{2k}$. Let $\mathscr{D}_n$ denote the subgroup of diagonal matrices in $GL(n, \mathbb{C})$; then a Cartan subgroup of the complexification of $SO^*(2k)$ can be identified with $\mathscr{D}_n$. Also a maximal compact subgroup of $SO^*(2k)$ can be identified with $U(k)$.

Let (G', G) be a pair of reductive subgroups in some real symplectic group, then (G', G) forms a *reductive dual pair* if G' is the centralizer of G and vice versa (See [Ho1], [MQ]). In this paper we shall consider the cases where G' is either the group $SO^*(2n)$ or the real symplectic group $Sp(2n, \mathbb{R})$. These dual pairs can be concretely realized as follows:

Let $\mathbb{C}^{n\times N}$ denote the vector space of all $n \times N$ complex matrices, with $N = 2k$ if $G' = SO^*(2n)$. Define a Gaussian measure $d\mu$ on $\mathbb{C}^{n\times N}$ by

$$d\mu(Z) = \pi^{-nN}\exp[-tr(ZZ^*)]dZ, \qquad Z \in \mathbb{C}^{n\times N},\ Z^* = \bar{Z}^T,$$

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where dZ denotes the Lebesgue measure on $\mathbb{C}^{n\times N}$. Let $\mathscr{F}(\mathbb{C}^{n\times N})$ denote the Fock space of all holomorphic square-integrable functions with respect to the measure $d\mu$. Then the groups $Sp(2k, \mathbb{C})$ and $SO(N, \mathbb{C})$ act on $\mathscr{F}(\mathbb{C}^{n\times N})$ by right translations and their Lie algebras act on $\mathscr{F}(\mathbb{C}^{n\times N})$ as differential operators (the infinitesimal operators of the group actions). The complexifications of the Lie algebra of G' $(= SO^*(2n)$ or $Sp(2n, \mathbb{R}))$ acts as differential operators on $\mathscr{F}(\mathbb{C}^{n\times N})$ that commute with the infinitesimal operators above (cf. [MQ]) so that in Howe's classification the dual (or complementary) groups G of G' $(= SO^*(2n)$ or $Sp(2n, \mathbb{R}))$ are $Sp(N)$ and $SO(N)$, respectively.

In general, the computation of the multiplicity of a weight of a character of an element of the discrete series of a linear connected reductive group is quite difficult (see, e.g., [Sc]); this can be done in two steps, for example, by combining the K spectrum formula of Hecht and Schmid (see [HS]) with Kostant's formula or Freudenthal's formula (see, e.g., [Hu]). Our main goal in this paper is to prove a computationally effective formula for the multiplicity of a weight of the character of an element of the holomorphic discrete series of $SO^*(2n)$. This will be done in the next section in connection with the explicit decomposition of an n-fold tensor product of irreducible "symmetric" representations of the dual group $Sp(N)$ of $SO^*(2n)$. It turns out that this formula can be generalized to an interesting abstract reciprocity type theorem for the dual pairs $(SO^*(2n), Sp(N))$ and $(Sp(2n, \mathbb{R}), SO(N))$, although it is not computationally as attractive as the particular case above. This will be done in the last section.

2. A multiplicity formula of a weight in a character of the holomorphic discrete series of $SO^*(2n)$

We retain the notation introduction in §1. If $(M) = (M_1, \dots, M_n)$ is an n-tuple of nonnegative integers, we define a holomorphic character $\xi^{(M)} : \mathscr{D}_n \to \mathbb{C}^*$ by

$$\xi^{(M)}(d) = d_{11}^{M_1} \cdots d_{nn}^{M_n}, \qquad d = \begin{pmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix} \in \mathscr{D}_n .$$

A polynomial function $p : \mathbb{C}^{n\times N} \to \mathbb{C}$ is said to *transform covariantly with respect to* $\xi^{(M)}$ if

$$p(dZ) = \xi^{(M)}(d)p(Z), \qquad (d, Z) \in \mathscr{D}_n \times \mathbb{C}^{n\times N} .$$

Let $\mathscr{P}^{(M)}(\mathbb{C}^{n\times N})$ denote the subspace of $\mathscr{F}(\mathbb{C}^{n\times N})$ of all polynomial functions which transform covariantly with respect to $\xi^{(M)}$. Then according to Theorem 2.7 of [KT] the $GL(N, \mathbb{C})$-module $\mathscr{P}^{(M)}(\mathbb{C}^{n\times N})$ is isomorphic to the $GL(N, \mathbb{C})$ Kronecker product

$$\underbrace{V^{(M_1)} \otimes \cdots \otimes V^{(M_n)}}_{n}$$

where $V^{(M_i)}$, $1 \le i \le n$, is a concrete realization (cf. [KT]) of an irreducible *symmetric* (*tensor*) *representation* of $GL(N, \mathbb{C})$ (see [We], p. 129) of signature $(M_i, 0, \dots, 0)$. Now it is easy to show (see [TT1]) that an $Sp(N, \mathbb{C})$-module $V^{(M_i)}$ remains irreducible and hence $V^{(M_i)}$ may be considered as a concrete realization of an irreducible *symmetric representation* of $Sp(N, \mathbb{C})$ (or equivalently $Sp(N)$) of signature

$$\underbrace{(M_i, 0, \dots, 0)}_{k}$$

where $k = \frac{N}{2}$. It follows that the subspace $\mathscr{P}^{(M)}(\mathbb{C}^{n\times N})$ may be viewed as a representation space of an n-fold tensor product of irreducible symmetric representations of $Sp(N)$ (or $Sp(N, \mathbb{C})$). An explicit decomposition of this tensor product is given in [Le]. Recall that every irreducible holomorphically induced representation of $Sp(N, \mathbb{C})$ (and hence every irreducible unitary representation of its compact real form $Sp(N)$) can be characterized by a $k = \frac{N}{2}$-tuple of nonnegative integers $(m) = (m_1, \dots, m_k)$, called the *signature* of the representation, which satisfy the dominant condition $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$. Now according to Harish-Chandra's realization of the holomorphic discrete series (see, e.g., [Kn], Theorem 6.6 and Theorem 9.20) a discrete series π_λ, where λ is a *nonsingular* Harish-Chandra parameter, of a linear connected semisimple Lie group G' may also be characterized by a K'-type (K' a maximal compact subgroup of G') Blattner parameter Λ: the lowest highest weight (minimal) K'-type. In the case that $G' = SO^*(2n)$, the dual of $Sp(N)$ under the actions above on $\mathscr{F}(\mathbb{C}^{n\times N})$, K' can be taken to be $U(n)$ and the Blattner parameter Λ may be labelled by an n-tuple of integers (m), similarly as in the case of the group $Sp(N)$, which for the sake of conciseness we shall also call the *signature* of π_λ. In the next theorem we let $k = n$ and hence $N = 2n$.

Theorem 2.1. *The dimension of the weight space indexed by $(M_1, \dots, M_n)$ in the irreducible representation of $SO^*(2n)$ with signature $(m) = (m_1, \dots, m_n)$ is equal to the frequency of occurrence of the irreducible representation of $Sp(2n, \mathbb{C})$ with signature $(m_1, \dots, m_n)$ in the n-fold tensor product representation of $Sp(2n, \mathbb{C})$ with signature*

$$(M_1, 0, \dots, 0) \otimes \cdots \otimes (M_n, 0, \dots, 0).$$

Proof. Fix a weight space indexed by an n-tuple $(M) \equiv (M_1, \dots, M_n)$ of positive integers of a holomorphic discrete series π_λ of $SO^*(2n)$ with signature (m). Then the weight indexed by (M) corresponds to the holomorphic character $\xi^{(M)} : \mathscr{D}_n \to \mathbb{C}^*$ and according to [KT] the dual of $(\mathscr{D}_n; (M))$ is $(GL(2n, \mathbb{C}) \times \cdots \times GL(2n, \mathbb{C}); (M_1) \otimes \cdots \otimes (M_n))$. From [Ho1] it follows that the dual of $(SO^*(2n); (m))$ is $(Sp(2n); (m))$ where (m) is also the signature of an irreducible holomorphically induced representation $R^{(m)}$ of $Sp(2n, \mathbb{C})$. Now let Λ' be the highest weight of a K' type $\pi_{\lambda|_{U(n)}}$; then Λ' corresponds to the signature $(\ell) = (\ell_1, \dots, \ell_n)$ of an irreducible holomorphically induced representation of $GL(n, \mathbb{C})$ with $\ell_1 \geq \cdots \geq \ell_n \geq 0$ (recall that by Theorem 9.20 of [Kn] $\Lambda' = \Lambda + \sum_{\alpha\in\Delta^+} n_\alpha \alpha$ for integers $n_\alpha \geq \alpha$). From [KT] the dual of $(GL(n, \mathbb{C}); (\ell))$ is $(GL(2n, \mathbb{C}), (\ell))$, where by abuse of language we denote by the same symbol (ℓ) the signature

$$\underbrace{(\ell_1, \dots, \ell_n, 0, \dots, 0)}_{2n}$$

of an irreducible holomorphically induced representation of $GL(2n, \mathbb{C})$. According to Corollary 2.9(a) of [KT] the dimension of the weight space indexed by (M) in the $GL(n, \mathbb{C})$-submodule of $\mathscr{P}^{(M)}(\mathbb{C}^{n\times N})$ of signature (ℓ) is equal to the frequency of occurrence of the irreducible representation of $GL(2n, \mathbb{C})$ with signature (ℓ) in $\mathscr{P}^{(M)}(\mathbb{C}^{n\times N})$. By Proposition 3.1 of [Ho1] the multiplicity of the K'-type representation labelled by Λ' (or (ℓ)) in the restriction $\pi_\lambda|_{U(n)}$ is equal to the multiplicity of the irreducible representation $R^{(m)}$ of $Sp(2n, \mathbb{C})$ in the restriction to $Sp(2n, \mathbb{C})$ of the irreducible representation of $GL(2n, \mathbb{C})$ with signature (ℓ). By letting Λ' vary over the K'-spectrum of the holomorphic discrete series π_λ and by summing over all appropriate multiplicities we can achieve the proof of the theorem. □

Remark 2.2. (i) A priori one does not know that the multiplicity of the weight (M) in the holomorphic discrete series with signature (m) is finite. Theorem 2.1 confirms that this is indeed always the case.

(ii) As we pointed out earlier the calculation of the multiplicity of weights, even in the case of the classical groups, is quite complicated (see, e.g., [DG] and [Ko]). In fact, using Eqs. (59) and (62) of [DG] one can, in principle, verify that the multiplicity of the representation $R^{(m)}$ of $Sp(2n, \mathbb{C})$ in the restriction to $Sp(2n, \mathbb{C})$ of the irreducible representation of $GL(2n, \mathbb{C})$ with signature (ℓ) is equal to the multiplicity of the K'-type representation labelled by (ℓ) in the restriction $\pi_\lambda|_{U(n)}$ as given by the formula in Theorem (1.3) of [HS]. The following formula which is analogous to a formula by H. Weyl for the general linear group allows us to compute easily the multiplicity of an irreducible representation of $Sp(2n, \mathbb{C})$ that occurs in the tensor product $\Lambda^{(m_1, \dots, m_n)} \otimes \Lambda^{(M, 0, \dots, 0)}$.

Proposition 2.3. *If $(m_1, \dots, m_n)$ is the signature of an irreducible holomorphic representation of $Sp(2n, \mathbb{C})$ and if $(M, 0, \dots, 0)$ is the signature of another irreducible holomorphic symmetric representation of $Sp(2n, \mathbb{C})$, then we have the following spectral formula:*

$$\begin{aligned}&(m_1, \dots, m_n) \otimes (M, 0, \dots, 0)\\ &\quad = \sum_{s=1, \dots, n} (m_1 + \mu_1 - \mu_{2n}, \dots, m_s + \mu_s - \mu_{2n-(s-1)}, \dots, m_n + \mu_n - \mu_{n+1})\end{aligned}$$

where the integers μ are subject to the following conditions:

$$(2.1)\qquad \begin{cases} \mu_1 + \cdots + \mu_{2n} = M, \\ 0 \le \mu_i \le m_{i-1} - m_i - \mu_{2n-(i-1)} + \mu_{2n-(i-2)}, \qquad 2 \le i \le n; \\ 0 \le \mu_{2n-j} \le m_{j+1} - m_{j+2}, \qquad 0 \le j \le n-2, \ 0 \le \mu_{n+1} \le m_n. \end{cases}$$

This formula can be derived using a result in [Li] (for details see [Le]); a proof of the particular case $(m, 0, \dots, 0) \otimes (M, 0, \dots, 0)$ can be found in [TT2]. By applying this formula n times and by following the procedure given in [KT] we thus obtain an explicit decomposition of $V^{(M_1, 0, \dots, 0)} \otimes \cdots \otimes V^{(M_n, 0, \dots, 0)}$ (see [Le] for details). In (2.1) if we set $\nu_s = \mu_s - \mu_{2n-(s-1)}$, $1 \le s \le n$, then the condition $\nu_1 + \cdots + \nu_n = M$ implies $\mu_1 = \nu_1, \dots, \mu_n = \nu_n$ and

$\mu_{n+1} = \cdots = \mu_{2n} = 0$. Equation (2.1) can then be written as

$$(m_1, \ldots, m_n) \otimes (M, 0, \ldots, 0)$$
$$= \sum_{\substack{0 \le \nu_{s+1} \le \nu_s, 1 \le s \le n-1 \\ \nu_1 + \cdots + \nu_n = M}} (m_1 + \nu_1, \ldots, m_s + \nu_s, \ldots, m_n + \nu_n) \tag{2.2}$$
$$+ \sum_{0 \le \nu_1 + \cdots + \nu_n < M} (m_1 + \nu_1, \ldots, m_n + v_n).$$

But the corresponding Weyl formula for $GL(n, \mathbb{C})$ is

$$(m_1, \ldots, m_n) \otimes (M, 0, \ldots, 0)$$
$$= \sum_{\substack{0 \le \nu_{s+1} \le \nu_s, 1 \le s \le n-1 \\ \nu_1 + \cdots + \nu_n = M}} (m_1 + \nu_1, \ldots, m_s + \nu_s, \ldots, m_n + \nu_n) \tag{2.3}$$

(see [Ze, p. 234]). By applying both formulae (2.2) and (2.3) n times and by comparing their spectra we can obtain the following interesting particular case of Theorem 2.1.

Corollary 2.4. *Let $(m_1, \ldots, m_n)$ and $(M_1, \ldots, M_n)$ be two n-tuples of integers such that $m_1 + \cdots + m_n = M_1 + \cdots + M_n$ and $m_1 \ge m_2 \ge \cdots \ge m_n \ge 0$, $M_i \ge 0, 1 \ge i \ge n$. Then the multiplicity of the weight $(M_1, \ldots, M_n)$ in the irreducible character of $SO^*(2n)$ with signature $(m_1, \ldots, m_n)$ is equal to the multiplicity of the irreducible representation of $GL(n, \mathbb{C})$ with signature $(m_1, \ldots, m_n)$ in the n-fold tensor product representation of $GL(n, \mathbb{C})$ with signature $(M_1, 0, \ldots, 0) \otimes \cdots \otimes (M_n, 0, \ldots, 0)$.*

Note that in practice the multiplicity on an irreducible representation of $Sp(2n, \mathbb{C})$ (resp. $GL(n, \mathbb{C})$) in an n-fold tensor product of $Sp(2n, \mathbb{C})$ (resp. $GL(n, \mathbb{C})$) with signature $(M_1, 0, \ldots, 0) \otimes \cdots \otimes (M_n, 0, \ldots, 0)$ can easily be obtained with the help of a computer program using formula (2.1) (resp. (2.3)) n times. Corollary 2.4 can be formulated in terms of Young diagrams (see Kostra formula (6.4;6) [DL, p. 104]). In light of this formula one can expect a simple description of Theorem 2.1 in terms of Young diagrams.

3. A reciprocity theorem for the dual pairs $(SO^*(2n), Sp(N))$ and $(Sp(2n, \mathbb{R}), SO(N))$

We retain the notation introduced in §1. Recall that every irreducible holomorphically induced representation of $SO(N, \mathbb{C})$ (and hence every irreducible unitary representation of its compact real form $SO(N)$) can be parametrized by an $[\frac{N}{2}]$-tuple of nonnegative integers $(m) = (m_1, \ldots, m_{[\frac{N}{2}]})$, where $[\frac{N}{2}]$ denotes the integral part of $\frac{N}{2}$ and where the m_i's, $1 \le i \le [\frac{N}{2}]$, satisfy the dominant conditions

$$m_1 \ge \cdots \ge m_{[\frac{N}{2}]-1} \ge |m_{[\frac{N}{2}]}| ,$$

if N is even and $m_1 \ge \cdots \ge m_{[\frac{N}{2}]} \ge 0$ if N is odd. The $[\frac{N}{2}]$-tuple (m) is called the signature of the representation. In this paper if N is even, we only consider the case $m_{N/2} \ge 0$. Similarly, as in the case of $SO^*(2n)$, a holomorphic discrete series of $Sp(2n, \mathbb{R})$ may be parametrized by an n-tuple of integers (m) which we shall again call its signature.

In the following theorem G denotes either $Sp(N)$ or $SO(N)$ and G' denotes either $SO^*(2n)$ or $Sp(2n, \mathbb{R})$.

Theorem 3.1. *Let n and N be positive integers. Let $(p_1, \ldots, p_n)$ be a partition of n, and for each $i = 1, \ldots, r$ let $(M_1^i, \ldots, M_{p_i}^i)$ be a p_i-tuple of integers such that $M_1^i \geq \cdots \geq M_{p_i}^i \geq 0$. Let $H' \equiv U(p_1) \times \cdots \times U(p_r)$ be the direct product of the unitary groups $U(p_i)$, $1 \leq i \leq r$.*

(i) *If $[\frac{N}{2}] \geq n$, then the multiplicity of the representation of H' with signature $\bigotimes_{i=1,\ldots,r}(M_1^i, \ldots, M_{p_i}^i)$ in the restriction to the subgroup H' of the irreducible representation of G' with signature $(m_1, \ldots, m_n)$ is equal to the frequency of occurrence of the irreducible representation of G with signature $(m_1, \ldots, m_n, 0, \ldots, 0)$ of G in the restriction to G of the r-fold tensor product representation of $\underbrace{U(N) \times \cdots \times U(N)}_{r} \equiv H$ with signature $(M_1^1, \ldots, M_{p_1}^1, 0, \ldots, 0) \otimes \cdots \otimes (M_1^r, \ldots, M_{p_r}^r, 0, \ldots, 0)$.*

(ii) *If $N \geq n > [\frac{N}{2}]$, then the multiplicity of the representation of H' with signature $\bigotimes_{i=1,\ldots,r}(M_1^i, \ldots, M_{p_i}^i)$ in the restriction to H' of the irreducible representation of G' with signature $(m_1, \ldots, m_{[\frac{N}{2}]}, 0, \ldots, 0)$ is equal to the frequency of occurrence of the irreducible representation of G with signature $(m_1, \ldots, m_{[\frac{N}{2}]})$ in the restriction to G of the r-fold tensor product representation of $\underbrace{U(N) \times \cdots \times U(N)}_{r}$ with signature $(M_1^1, \ldots, M_{p_1}^1, 0, \ldots, 0) \otimes \cdots \otimes (M_{p_1}^r, \ldots, M_{p_r}^r, 0, \ldots, 0)$.*

Proof. (i) Let $(m) \equiv (m_1, \ldots, m_n)$ denote the signature of a holomorphic discrete series π_λ of G'; from [Ho1] there exists a corresponding irreducible holomorphically induced representation of the complexification of G with signature (m). As mentioned before, a maximal compact subgroup of G' is K' which is isomorphic to $U(n)$. Let (μ) denote the signature on an irreducible representation of K' that occurs in $\pi_\lambda|_{K'}$. From [KT] there is a corresponding irreducible representation of $K \approx U(N)$ with the same signature (μ). Now let $\bigotimes_{i=1,\ldots,r}(M_1^i, \ldots, M_{p_i}^i)$ be the signature of an irreducible representation of H'; then again from [KT] there is a corresponding representation of H with signature $(M_1^1, \ldots, M_{p_1}^1, 0, \ldots, 0) \otimes \cdots \otimes (M_1^r, \ldots, M_{p_r}^r, 0, \ldots, 0)$. By [Ho1] and the Weyl's "unitarian trick" we have the following equality:

$$\#[(\mu)_{K'}, (m)|_{K'}^{G'}] = \#[(m)_{G'}, (\mu)|_G^K] \tag{3.1}$$

where $\#[(\mu)_{K'}, (m)|_{K'}^{G'}]$ denotes the multiplicity of $(\mu)_{K'}$ in the restriction of $(m)_{G'}$ to K', etc. By a Proposition in §2 of [Ho2] we also have the equality

$$\begin{aligned}\#&\left[\left(\bigotimes_{i=1,\ldots,r}(M_1^i, \ldots, M_{p_i}^i)\right)_{H'}, (\mu)|_{H'}^{K'}\right] \\ &= \#[(\mu)_K, ((M_1^1, \ldots, M_{p_1}^1, 0, \ldots, 0) \\ &\qquad \otimes \cdots \otimes (M_1^r, \ldots, M_{p_r}^r, 0, \ldots, 0))|_K^H].\end{aligned} \tag{3.2}$$

Letting μ vary over the K'-spectrum of π_λ we obtain from (3.1) and (3.2)

$$\begin{aligned}
\#&\left[\left(\bigotimes_{i=1,\dots,r}(M_1^i,\dots,M_{p_i}^i)\right)_{H'},(m)|_{H'}^{G'}\right]\\
&=\sum_{(\mu)}\#\left[\left(\bigotimes_{i=1,\dots,r}(M_1^i,\dots,M_{p_i}^i)\right)_{H'},(\mu)|_{H'}^{K'}\right]\#[(\mu)_{K'},(m)|_{K'}^{G'}]\\
&=\sum_{(\mu)}\#[(m)_G,(\mu)|_G^K]\#[(\mu)_K,((M_1^1,\dots,M_{p_1}^1,0,\dots,0)\\
&\qquad\qquad\otimes\cdots\otimes(M_1^r,\dots,M_{p_r}^r,0,\dots,0))|_K^H]\\
&=\#[(m)_G,(M_1^1,\dots,M_{p_1}^1,0,\dots,0)\\
&\qquad\otimes\cdots\otimes(M_1^r,\dots,M_{p_r}^r,0,\dots,0)|_G^H]
\end{aligned}$$

which is exactly the desired equality

(ii) The proof of this part is similar to that of part (i), but one must impose the additional condition $n \le N$ because the K'-spectrum of π_λ can contain a signature of the form $(\mu_1,\dots,\mu_n)$ which corresponds to an irreducible representation of K if and only if $n \le N$. □

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References

[DG] R. M. Delaney and B. Gruber, *Inner and restriction multiplicity of classical groups*, J. Math. Phys. **10** (1965), 252–265.

[DL] D. Littlewood, *The theory of group characters and matrix representations of groups*, second ed., Clarendon Press, Oxford, 1950.

[HS] H. Hecht and W. Schmid, *A proof of Blattner's conjecture*, Invent. Math. **31** (1975), 129–154.

[Ho1] R. Howe, *Reciprocity laws in the theory of dual pairs*, Representation Theory of Reductive Groups (P. C. Trombi, ed.), Progr. Math., vol. 40, Birkhauser, Boston, MA, 1983, pp. 159–175.

[Ho2] ——, *$(GL(n), GL(m))$-duality and symmetric plethysm*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), 85–109.

[Hu] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, Berlin, Heidelberg, and New York, 1972.

[KT] W. H. Klink and T. Ton-That, *n-fold tensor products of $GL(N, \mathbb{C})$ and decomposition of Fock spaces*, J. Funct. Anal. **84** (1989), 1–18.

[Ko] K. Koike, *On new multiplicity formulas of weights of representations of classical groups*, J. Algebra **107** (1987), 512–533.

[Kn] A. Knapp, *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton, NJ, 1986.

[Le] E. Y. Leung, *On resolving the multiplicity of tensor products of irreducible representations of symplectic groups*, Ph.D. Thesis, University of Iowa, Iowa City, Iowa, 1993.

[Li] P. Littelmann, *A generalization of Littlewood-Richardson rule*, J. Algebra **130** (1990), 328–368.

[MQ] M. Moshinsky and C. Quesne, *Noninvariance groups in the second quantization picture and their applications*, J. Math. Phys. **11** (1970), 1631–1639.

[Sc] W. Schmid, *On the characters of discrete series: the Hermitian symmetric case*, Invent. Math. **30** (1975), 47–144.

[TT1] T. Ton-That, *Symplectic Stiefel harmonics and holomorphic representations of symplectic groups*, Trans. Amer. Math. Soc. **232** (1977), 265–277.

[TT2] ——, *Explicit decomposition of tensor products of certain analytic representations of symplectic groups*, J. Math. Phys. **19** (1978), 2514–2519.

[We] H. Weyl, *The classical groups, their invariants and representations*, second ed., Princeton Univ. Press, Princeton, NJ, 1946.

[Ze] D. P. Želobenko, *Compact Lie groups and their representations*, Amer. Math. Soc., Providence, RI, 1973.

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