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A CHARACTERIZATION OF QUASINORMABLE KÖTHE SEQUENCE SPACES

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ABSTRACT. Let E be a quasinormable Fréchet space. We prove that every quotient map $q: E \to X$ with X Banach lifts bounded sets. Moreover, we show that this property characterizes the quasinormability of E in case that E is a Köthe sequence space of order p, $1 \leq p < \infty$ or $p = 0$.

The class of quasinormable locally convex spaces was introduced and studied by Grothendieck in [9] and has recently received much attention, particularly in the context of Fréchet spaces (see [1, 2, 13]). We recall that a Fréchet space E with a decreasing zero-neighborhood basis $(U_n)_{n\in\mathbb{N}}$ is *quasinormable* if there is a bounded set B in E such that

$$\forall n \in \mathbb{N}\ \exists m > n, \quad \forall \varepsilon > 0\ \exists \lambda > 0, \qquad U_m \subset \lambda B + \varepsilon U_n$$

(see [11, 10.7.2]). In this note we prove two results on the structure of quasinormable Fréchet spaces. In the first section we show that *every quotient map q from a quasinormable Fréchet space E onto a Banach space X lifts bounded sets* (*i.e., given a bounded set C in X there is a bounded set B in E such that C is contained in* $q(B)$). We also prove that this property actually characterizes the quasinormability in the class of Köthe sequence spaces. In §2 we show that *a quasinormable Fréchet space E without copies of l_1 can be written as a projective limit of a sequence of Banach spaces without copies of l_1. In particular no quotient of E contains a copy of l_1.*

Our notation is standard. We refer the reader to [12, 14].

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Before stating our main result we recall the celebrated example of Köthe of a Fréchet-Montel sequence space λ_1 having l_1 as a quotient [12, 31.5]. Denote by q the quotient map; then for any bounded set C in λ_1, we have that $q(C)$ is precompact and, consequently, cannot contain the unit ball of l_1. In particular, q does not lift bounded sets. With our main result we show that lacking the quasinormability seems to be the main reason for such a pathology.

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1. **Theorem.** *Let E be a quasinormable Fréchet space, and let L be a closed subspace of E such that E/L is normable. Then the canonical quotient map $q: E \to E/L$ lifts bounded sets.*

Proof. It is enough to show that there is a bounded set B in E such that $q(B)$ is a zero-neighborhood in E/L. Indeed, by quasinormability there is an absolutely convex bounded set B in E such that

$$(1) \qquad \forall n \in \mathbb{N}\ \exists m > n, \quad \forall \varepsilon > 0\ \exists \lambda > 0, \qquad U_m \subset \lambda B + \varepsilon U_n.$$

Let V denote the unit ball of E/L, and fix $n \in \mathbb{N}$ such that $q(U_n) \subset V$. Then we choose $m > n$ as in (1). Since q is open, there is $\rho > 0$ such that $\rho V \subset q(U_m)$. Given $\varepsilon = (2\rho)^{-1}$ we apply q in (1) to get $V \subset \lambda q(B) + \frac{1}{2}V$ for some $\lambda > 0$. This inclusion implies $V \subset 2\lambda\overline{q(B)}$. Hence $\overline{q(B)}$ is a zero-neighborhood in X. To finish, the same technique of the Banach-Schauder theorem [12, 15, 12.(2)] allows one to remove the closure and obtain that $q(B)$ is a zero-neighborhood. We can also apply the following relevant result recently proved by Bonet and Dierolf [4]: *let E be a Fréchet space, let L be a closed subspace of E and $q: E \to E/L$ the quotient map. If q lifts bounded sets with closure (i.e., for every K bounded in E/L there is a bounded set A in E such that $K \subset \overline{q(A)}$), then q lifts bounded sets.* □

We provide two consequences of Theorem 1. The first should be compared with [17, 1.5 and 1.6]. The second extends to quasinormable Fréchet spaces a well-known result from the theory of Banach spaces.

2. **Corollary.** (i) *Let $0 \to L \to E \xrightarrow{q} G \to 0$ be an exact sequence where E is Fréchet and G is a Schwartz or a Banach space; then L is quasinormable if and only if E is quasinormable.*

(ii) *Let $0 \to L \to E \xrightarrow{\psi} l_1(I) \to 0$ be an exact sequence. If E is Fréchet quasinormable, then ψ has a right inverse. In particular, if a quasinormable Fréchet space has a quotient isomorphic to $l_1(I)$, then it also has a complemented copy of $l_1(I)$.*

Proof. (i) If L and G are quasinormable, then E is quasinormable by the positive solution of the three-space problem for quasinormable Fréchet spaces (see [15]). Conversely, assume that G is either Schwartz or Banach and E is quasinormable. In the Schwartz case by compactness (see [12, 22.2.7]) and in the Banach case, by our Theorem 1 it follows that $q: E \to G$ is a quotient map that lifts bounded sets. Then, the result of Merzon (see [6]) implies that $\operatorname{Ker} q$, i.e., L, is also quasinormable.

(ii) By Theorem 1 there is a bounded set $(x_i)_{i \in I}$ in E such that $\psi(x_i) = e_i$ $(i \in I)$ where e_i denotes the ith unit vector basis. Since $(x_i)_{i \in I}$ is bounded, it is readily checked that the map

$$\imath : l_1(I) \to E, \qquad (\alpha_i)_{i \in I} \to \sum_{i \in I} \alpha_i x_i,$$

is a well-defined and continuous linear mapping. It is also clear that $\psi \circ \imath$ is the identity of $l_1(I)$. □

3. *Remark.* Compare with 2(ii) the following statement which follows from results in [5] and [16]. *If $0 \to L \to E \xrightarrow{\psi} l_1(I)^{\mathbb{N}} \to 0$ is an exact sequence with*

L (*and hence* E) *a quojection* (*i.e., a countable projective limit of surjective operators*), *then* ψ *has a right inverse.* We recall that a quojection is a very particular case of a quasinormable Fréchet space.

To finish this section we shall prove that lifting bounded sets from Banach quotients characterizes the quasinormability in the class of Köthe sequence spaces. We begin with some definitions.

4. **Definition** [1]. Let I be an index set. A *Köthe matrix* on I is a sequence of maps $A = (a_k(i))_{k\in\mathbb{N}}$, $a_k: I \to \mathbb{R}$ $(k \in \mathbb{N})$, satisfying: (i) $0 \le a_k(i) \le a_{k+1}(i)$, $k \in \mathbb{N}$, $i \in I$, and (ii) $\forall i \in I$ $\exists k \in \mathbb{N}$, $a_k(i) > 0$. Given a Köthe matrix A and given $p \in \mathbb{R}$, $1 \le p < \infty$, or $p = 0$, we define the *Köthe sequence space of order* p as

$$\lambda_p(I, A) := \left\{ (x_i) \in \mathbb{K}^I;\ \|(x_i)\|_k := \left(\sum_{i\in I} |x_i|^p a_k(i) \right)^{1/p} < \infty,\ k \in \mathbb{N} \right\},$$

$$1 \le p < \infty;$$

$$\lambda_0(I, A) := \left\{ (x_i) \in \mathbb{K}^I;\ \lim_{i\in I} |x_i| a_k(i) = 0,\ \|(x_i)\|_k := \sup_{i\in I} |x_i| a_k(i) \right\}.$$

$\lambda_p(I, A)$ is a Fréchet space when endowed with the topology defined by the increasing sequence of seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ (see [1]) for more details).

The next statement follows from the characterization given in [1]. *A Köthe sequence space* $\lambda_p(I, A)$ *is quasinormable if and only if for every countable set* $J \subset I$ *the sectional subspace*

$$\lambda_p(J, A) := \{(x_i) \in \lambda_p(I, A);\ x_i = 0\ \forall i \notin J\}$$

is quasinormable. This is the reason why we assume that I is always either $\mathbb{N}$ or $\mathbb{N} \times \mathbb{N}$, and we shall write only $\lambda_p(A)$ instead of $\lambda_p(I, A)$.

5. **Theorem.** *Let* $\lambda_p(A)$ *be a Köthe sequence space, with* $1 \le p < \infty$ *or* $p = 0$. *Assume that every quotient map* $q: \lambda_p(A) \to X$, *with* X *Banach, lifts bounded sets. Then* $\lambda_p(A)$ *is quasinormable.*

Proof. We shall assume that $\lambda_p(A)$ is not quasinormable, and we shall construct a quotient of $\lambda_p(A)$, isomorphic to l_p (isomorphic to c_0 in case $p = 0$) and such that the quotient map does not lift bounded sets.

Let A be a Köthe matrix such that $\lambda_p(A)$ is not quasinormable. It is proved in [3] that $\lambda_p(A)$ has a quotient $\lambda_p(B)$ where the Köthe matrix $B = (b_k(i, j))_{k\in\mathbb{N}}$ satisfies:

(a) $b_1(i, j) = 1$, $\forall i, j \in \mathbb{N}$, $b_k(i, j) = b_1(i, j)$, $\forall i \ge k$.
(b) $\lim_{j\to\infty} b_k(k-1, j) = \infty$, $\forall k \ge 2$.

For technical reasons we modify the weights defining $b_k^*(i, j) := i b_k(i, j)$. It is clear that $\lambda_p(B)$ is isomorphic to $\lambda_p(B^*)$, where $B^* := (b_k^*(i, j))_{k\in\mathbb{N}}$. We are done if we construct the quotient map that does not lift bounded sets for the space $\lambda_p(B^*)$. We have got the following conditions:

(a′) $b_1^*(i, j) = i$, $\forall i, j \in \mathbb{N}$; $b_k^*(i, j) = b_1^*(i, j)$, $\forall i \ge k$.
(b′) $\lim_{j\to\infty} b_k^*(k-1, j) = \infty$ $\forall k \ge 2$.

We proceed with the proof in three steps.

Step 1: *We construct the quotient.* As a consequence of (a′) we have,

$$c_k(j) := \inf\{b_k^*(i, j);\ i \in \mathbb{N}\} = \inf\{i b_k(i, j);\ i \in \mathbb{N}\} = \min\{i b_k(i, j);\ i \le k\}.$$

It now follows from [3, 3.1] (see also [18]) that $\lambda_p(C)$, with $C := (c_k(j))_{k\in\mathbb{N}}$, is a quotient of $\lambda_p(B^*)$.

Step 2: *We check that $\lambda_p(C)$ is normable.* It follows from (b) that

$$\lim_{j\to\infty} b_k(i, j) = \infty, \quad \forall i < k; \qquad b_k(k, j) = 1, \quad \forall j \in \mathbb{N}.$$

As a consequence $c_k(j)$ takes the value k for almost every j. Therefore for some $M_k > 0$ we get $c_k(j) \le M_k c_1(j)$ $(k \in \mathbb{N})$. This implies that the first seminorm $\|\cdot\|_1$ defines the topology of $\lambda_p(C)$.

Step 3: *We prove that the quotient map $\Pi : \lambda_p(B^*) \to \lambda_p(C)$, $\Pi((x_{ij})) := (\sum_{i=1}^{\infty} 2^{-i} x_{ij})_j$ does not lift bounded sets.* Denote by V the unit ball of the first seminorm in $\lambda_p(C)$. It is enough to check that V is not contained in $\Pi(L)$ for any bounded set L in $\lambda_p(B^*)$. Observe that $e_n \in V$ where e_n denotes, as usual, the nth unit vector basis. Take L as any bounded set in $\lambda_p(B^*)$. We will show that there is $j \in \mathbb{N}$ such that for every $x \in L$, the jth compound of $\Pi(x)$ is less than $\frac{1}{3}$. This in particular implies that $e_j \notin \Pi(L)$. We assume that $1 \le p < \infty$; the case $p = 0$ is similar.

Let L be bounded in $\lambda_p(B^*)$. Let us denote $M_k := \sup\{\|x\|_k ; x \in L\}$ and fix a positive integer $i_0 \ge (3M_1)^p$. Given any $x = (x_{ij}) \in L$ it follows by (a′) that

$$|x_{ij}| i^{1/p} \le \left(\sum_{i=1}^{\infty} |x_{ij}|^p b_1^*(i, j) \right)^{1/p} \le \|(x_{ij})\|_1 \le M_1, \quad \forall i, j \in \mathbb{N},$$

whence $|x_{ij}| \le \frac{1}{3}$, $\forall i \ge i_0$, $\forall x = (x_{ij}) \in L$. Analogously, given $i \in \mathbb{N}$ and $x = (x_{ij}) \in L$ we have

$$|x_{ij}| b_{i+1}^{1/p}(i, j) \le \|(x_{ij})\|_{i+1} \le M_{i+1}, \quad \forall j \in \mathbb{N}.$$

Thus by (b′) we can fix $j(i) \in \mathbb{N}$ such that $|x_{ij}| < \frac{1}{3}$ for every $j \ge j(i)$ and $x \in L$. To finish we take $j_0 = \max\{j(i) ; 1 \le i \le i_0 - 1\}$ and evaluate the j_0th compound of $\Pi(x)$ for any $x \in L$. By the above inequalities we get,

$$\left| \sum_{i=1}^{\infty} 2^{-i} x_{ij_0} \right| \le \sum_{i=1}^{i_0 - 1} 2^{-i} |x_{ij_0}| + \sum_{i=i_0}^{\infty} 2^{-i} |x_{ij_0}| < \tfrac{1}{3}.$$

This finishes the proof. □

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The following problem arises in several situations in Analysis:

(∗) *Let E be a Fréchet space with a certain property* (P). *When can E be written as a projective sequence of Banach spaces having the property* (P)?

It was proved by Grothendieck that a quasinormable reflexive Fréchet space E can be written as a projective sequence of reflexive Banach spaces [9]. Here we show a similar result if (P) is the property of having no copy of l_1.

6. **Theorem.** *Let E be a quasinormable Fréchet space without copies of l_1. Then E can be written as a projective limit of Banach spaces having no copy of l_1. In particular, no quotient of E contains a copy of l_1.*

Proof. Let us denote by E_n the local nth Banach space and by $I_n : E \to E_n$, $I_{n,m} : E_m \to E_n$, $n \leq m$, the canonical mappings. By quasinormability and by taking a subsequence of $(U_n)_{n\in\mathbb{N}}$ if it is necessary we may assume that there is an absolutely convex bounded set B in E satisfying

$$\forall n \in \mathbb{N}\ \forall \varepsilon > 0\ \exists \lambda > 0, \qquad U_{n+1} \subset \lambda B + \varepsilon U_n.$$

In particular, for every $n \in \mathbb{N}$, we have

$$\forall \lambda > 0\ \exists \varepsilon > 0, \qquad I_n(U_{n+1}) \subset \lambda I_n(B) + \varepsilon I_n(U_n). \tag{1}$$

Since E has no copy of l_1, B is weakly conditionally compact (w.c.c.) (i.e., every sequence in B has a weak Cauchy subsequence; e.g., see [7, Lemma 3]); therefore, $I_n(B)$ is w.c.c. for every $n \in \mathbb{N}$. By (1) and [8, p. 237] it follows that $I_n(U_{n+1})$ is w.c.c. Now it is readily checked that $I_{n,n+1}$ is a Rosenthal operator (i.e., it maps the unit ball of E_{n+1} into a w.c.c. set). It is a result of Weis [19] that every Rosenthal operator factorizes through a Banach space having no copy of l_1; whence, the first assertion in this theorem follows.

Now let F be a quotient of E with quotient map q, and let us check that F has no copy of l_1. Indeed, take $V_n := q(U_n)$ $(n \in \mathbb{N})$ as a zero-neighborhood basis of F and denote by F_n, $n \in \mathbb{N}$, the corresponding local Banach spaces and by $J_n : F \to F_n$, $J_{n,m} : F_m \to F_n$ the canonical mappings. Then we have the following commutative diagram where the q_n's stand by canonical extensions:

$$\begin{array}{ccc} E_n & \xrightarrow{q_n} & F_n \\ {\scriptstyle I_{n,n+1}}\uparrow & & \uparrow{\scriptstyle J_{n,n+1}} \\ E_{n+1} & \xrightarrow{q_{n+1}} & F_{n+1} \end{array}$$

Since $I_{n,n+1}$ is a Rosenthal operator, $J_{n,n+1}$ is a Rosenthal operator too. Hence F can also be written as a projective limit of Banach spaces having no copy of l_1. This already implies that F has no copy of l_1. □

7. *Remark.* The result above does not hold for a general Fréchet space. Once more, the classical Montel, non-Schwartz, Köthe sequence space λ_1 [12] does not contain any copy of l_1, but it cannot be written as a projective sequence of Banach spaces without copies of l_1. This can be deduced from the fact that every Rosenthal operator from l_1 into l_1 is compact. It could also be derived from the second part of the proof of Theorem 6 where quasinormability has not been used.

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