OPERATORS WITH COMPLEX GAUSSIAN KERNELS: BOUNDEDNESS PROPERTIES

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Abstract. Boundedness properties are stated for some operators from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, $1 \leq p, q \leq \infty$, with complex Gaussian kernels. Their contraction properties are also analysed.

1. Introduction

In this paper we will study boundedness properties for the general complex Gaussian operator in dimension one,

$$\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma}(x) = \int_{-\infty}^{+\infty} \exp\{-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y\} \cdot f(y) \, dy,$$

$\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{C}, x \in \mathbb{R}$, from the space of complex-valued functions $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, $1 \leq p, q \leq \infty$, relative to the Lebesgue measure.

This subject was originally of interest in the context of Quantum Field Theory (see [1]). The complex Gaussian operator (1.1) has an intrinsic interest due to the basic role of the extended oscillator semigroup introduced by Howe [4] (see also Folland [3, Chapter 5]).

In his important paper [5], Lieb extends the operator (1.1) to $n$ dimensions and develops an extensive study of (1.1) in the context of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Moreover, he generalizes the results given by Epperson in §2 of [2] for (1.1). Lieb stated that for the nondegenerate case, that is, $(\text{Re} \delta)^2 < (\text{Re} \beta) \cdot (\text{Re} \varepsilon)$, $\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma}$ has exactly one maximizer which is a centered Gaussian function $e^{s y^2}$, $s \in \mathbb{C}$. For the degenerate case, that is, $(\text{Re} \delta)^2 = (\text{Re} \beta) \cdot (\text{Re} \varepsilon)$, the question of the existence of a maximizer is a subtle one. This problem requires a complicated algebraic study, and precise conditions are not given there.

The purpose of our paper is to calculate the exact region of boundedness of (1.1) for each $1 \leq p, q \leq \infty$, in both degenerate and nondegenerate cases.

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The proof of this result follows a technique initiated by Weissler in Theorem 1 of [6]. Weissler stated the exact region of boundedness for the Hermite semigroup.

We conclude this paper by giving sufficient conditions in both degenerate and nondegenerate cases for the operator \( \mathcal{F}_{\beta, e, \delta, \xi, \gamma} \), \( \beta, e, \delta, \xi, \gamma \in \mathbb{R} \), to be a contraction over \( L^p(\mathbb{R}) \), \( 1 < p < \infty \), and a contraction as an operator from \( L^2(\mathbb{R}) \) into \( L^p(\mathbb{R}) \), \( 0 < p < \infty \). These results follow those given in Chapter 8 of [1].

Throughout this paper we will take square roots with positive real part.

2. Boundedness properties

For \( 1 < p < \infty \), let \( (I_p \gamma)(x) = \left(\frac{2\pi}{p}\right)^{1/2} \exp(-x^2/2p) \cdot f(x) \). For \( \gamma \in \mathbb{C} \), let \( (Q_\gamma \gamma)(x) = e^{\gamma x} \cdot f(x) \). For \( \gamma^* > 0 \), let \( (T_{\gamma^*} \gamma)(x) = f(\gamma^* x) \). Also, for \( \alpha^* \in \mathbb{C} \), let \( (M_{\alpha^*} \gamma)(x) = \exp(-\alpha^* x^2/2) \cdot f(x) \). The Gauss-Weierstrass semigroup on \( \mathbb{R} \) is given by

\[
(e^{\Delta f})(x) = (4\pi)^{-1/2} \cdot \int_{-\infty}^{+\infty} \exp[-(x-y)^2/4z] \cdot f(y) \, dy, \quad x \in \mathbb{R},
\]

where \( \text{Re} \, z \geq 0 \) (and \( z \neq 0 \)). Finally, we denote by \( \|\mathcal{F}_{\beta, e, \delta, \xi, \gamma}\|_{p,q} \) the norm of \( \mathcal{F}_{\beta, e, \delta, \xi, \gamma} \) as a map from \( L^p(\mathbb{R}) \) into \( L^q(\mathbb{R}) \), \( 1 \leq p, q \leq \infty \).

It is easy to check that for any \( \gamma^* > 0 \) and \( \text{Re} \, \delta > 0 \),

\[
(2.1) \quad \mathcal{F}_{\beta, e, \delta, \xi, \gamma} = (\pi \gamma^*/\delta)^{1/2} (2\pi)^{(1/2p)-(1/2q)} Q_\xi I_q^{-1} M_{\beta^*} T_{\gamma^*} e^{(\gamma^*/4\delta)\Delta} M_{\alpha^*} I_p Q_\gamma,
\]

where

\[
\alpha^* = 2\varepsilon - \frac{1}{p} - \frac{2\delta}{\gamma^*}, \quad \beta^* = 2\beta + \frac{1}{q} - 2\gamma^* \delta.
\]

We will denote by \( q' \) the exponent conjugate to \( q \).

Theorem 2.1. The following hold:

(i) If \( 1 \leq p \leq q \leq \infty \), \( \text{Re} \, \delta > 0 \), and \( \text{Re} \, \epsilon > 0 \), then \( \mathcal{F}_{\beta, e, \delta, \xi, \gamma} \) is bounded from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) if and only if

\[
(\text{Re} \, \epsilon) \cdot (\text{Re} \, \beta) \geq (\text{Re} \, \delta)^2.
\]

(ii) If \( 1 \leq q < p \leq \infty \), \( \text{Re} \, \delta > 0 \), and \( \text{Re} \, \epsilon > 0 \), then \( \mathcal{F}_{\beta, e, \delta, \xi, \gamma} \) is bounded from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) if and only if

\[
(\text{Re} \, \epsilon) \cdot (\text{Re} \, \beta) > (\text{Re} \, \delta)^2.
\]

(iii) If \( 1 \leq p \leq q \leq \infty \), \( \text{Re} \, \delta = 0 \), \( \frac{1}{q} - \frac{1}{2p} \leq \text{Re} \, \epsilon \leq \frac{1}{2} - \frac{1}{2p} \), and \( \text{Re} \, \beta \geq \text{Re} \, \epsilon + \frac{1}{2p} - \frac{1}{2q} \geq 0 \), then \( \mathcal{F}_{\beta, e, \delta, \xi, \gamma} \) is bounded from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) if and only if \( \text{Re} \, \epsilon \geq 0 \).

(iv) If \( 1 \leq p \leq q \leq \infty \), \( \text{Re} \, \delta = 0 \), \( \frac{1}{2p} - \frac{1}{2q} \leq \text{Re} \, \beta \leq \text{Re} \, \epsilon + \frac{1}{2p} - \frac{1}{2q} \), and \( \frac{1}{4} - \frac{1}{2q} \leq \text{Re} \, \beta \leq \frac{1}{2} - \frac{1}{2q} \), then \( \mathcal{F}_{\beta, e, \delta, \xi, \gamma} \) is bounded from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) if and only if \( \text{Re} \, \epsilon \geq 0 \) and \( \text{Re} \, \beta \geq 0 \).

Proof. Suppose first that \( \text{Re} \, \delta = 0 \), \( \frac{1}{4} - \frac{1}{2p} \leq \text{Re} \, \epsilon \leq \frac{1}{2} - \frac{1}{2p} \), \( \text{Re} \, \delta \geq \text{Re} \, \epsilon + \frac{1}{2p} - \frac{1}{2q} \geq 0 \), and \( 1 \leq p, q \leq \infty \). Now, if \( \text{Re} \, \epsilon \geq 0 \), \( \text{Re} \, \beta \geq 0 \), and \( m \) is a real number such that \( \text{Re} \, 2\epsilon + \frac{1}{m} = \frac{1}{m} \), we have \( 1 \leq m \leq 2 \), and the Hausdorff-Young inequality yields that \( e^{(\gamma^* /4\delta)\Delta} \) is bounded from \( L^m(\mathbb{R}) \) into \( L^{m'}(\mathbb{R}) \) \( (m' \) is the
exponent conjugate to \( m \). But from the hypothesis \( \mathcal{F}_{b_{-(1/2q), e+(1/2p), \delta, \xi, \gamma}} \) is bounded from \( L^m(\mathbb{R}, e^{-x^2/2} dx) \) into \( L^m(\mathbb{R}, e^{-x^2/2} dx) \). Since \( p \geq m \) and \( q \leq m' \), the operator \( \mathcal{F}_{b_{-(1/2q), e+(1/2p), \delta, \xi, \gamma}} \) is bounded from \( L^p(\mathbb{R}, e^{-x^2/2} dx) \) into \( L^q(\mathbb{R}, e^{-x^2/2} dx) \), \( 1 \leq p, q \leq \infty \), and therefore \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) is bounded from \( L^p(\mathbb{R}) \) into \( L^q(\mathbb{R}) \).

For the case \( \Re \delta = 0 \), \( \frac{1}{2p} - \frac{1}{2q'} \leq \Re \beta \leq \Re \varepsilon + \frac{1}{2p} - \frac{1}{2q} \), \( \frac{1}{2} - \frac{1}{2q'} \leq \Re \beta \leq \frac{1}{2} - \frac{1}{2q} \), and \( 1 \leq p, q \leq \infty \), the proof is similar.

Now assume \( \Re \delta > 0 \), so that \( \Re(\gamma^*/4\delta) > 0 \) and therefore, for \( 1 \leq p \leq q \leq \infty \), \( e^{(\gamma^*/4\delta)^A} \) is bounded from \( L^p(\mathbb{R}) \) into \( L^q(\mathbb{R}) \). Since \( \Re \varepsilon \geq 0 \), \( \Re \beta \geq 0 \), and \( (\Re \varepsilon) \cdot (\Re \beta) \geq (\Re \delta)^2 \), \( \gamma^* \) can be chosen so that \( \Re \alpha^* \geq 0 \) and \( \Re \beta^* \geq 0 \). It follows from (2.1) that \( \mathcal{F}_{b_{-(1/2q), e+(1/2p), \delta, \xi, \gamma}} \) is bounded from \( L^p(\mathbb{R}, e^{-x^2/2} dx) \) into \( L^q(\mathbb{R}, e^{-x^2/2} dx) \), \( 1 \leq p, q \leq \infty \), and hence \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) is bounded from \( L^p(\mathbb{R}) \) into \( L^q(\mathbb{R}) \).

For the case \( 1 < q < p < \infty \) and from the conditions \( \Re \varepsilon > 0 \), \( \Re \beta > 0 \), and \( (\Re \varepsilon) \cdot (\Re \beta) > (\Re \delta)^2 \), \( \gamma^* \) can be chosen so that \( \Re \alpha^* > 0 \) and \( \Re \beta^* > 0 \). Observing that \( M_{\beta^*} \) is a bounded map from \( L^p(\mathbb{R}) \) into \( L^q(\mathbb{R}) \) for \( \Re \beta^* > 0 \) and since \( e^{(\gamma^*/4\delta)^A} \) is bounded over \( L^p(\mathbb{R}) \), equality (2.1) implies that \( \mathcal{F}_{b_{-(1/2q), e+(1/2p), \delta, \xi, \gamma}} \) is bounded from \( L^p(\mathbb{R}, e^{-x^2/2} dx) \) into \( L^q(\mathbb{R}, e^{-x^2/2} dx) \), \( 1 < q < p < \infty \). Therefore, \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) is bounded from \( L^p(\mathbb{R}) \) into \( L^q(\mathbb{R}) \).

In order to prove the converse, suppose \( \| \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \|_{p, q} < \infty \). We will prove that \( \Re \varepsilon > 0 \), \( \Re \beta > 0 \), and \( (\Re \varepsilon) \cdot (\Re \beta) > (\Re \delta)^2 \) holds if \( q < p \). To this end, we need to calculate the action of \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) on an arbitrary Gaussian function \( g_s(y) = e^{sy}, s \in \mathbb{C}, y \in \mathbb{R} \). Then \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) can be computed for \( \Re s < \Re \varepsilon \) to obtain

\[
\mathcal{F}_{b_{, e, \delta, \xi, \gamma}} g_s(x) = \left( \frac{\pi}{e-s} \right)^{1/2} \exp \left( \frac{\delta^2 - \beta e + \beta s}{e-s} \right) \frac{\delta \gamma + \xi e - \xi s}{e-s} x + \frac{\gamma^2}{4(e-s)}, \quad x \in \mathbb{R}.
\]

We impose that \( g_s \in L^p(\mathbb{R}) \) so that \( \Re s < 0 \). Now, we want (2.2) to be in \( L^q(\mathbb{R}) \). With this purpose, let us consider the transformation \( L(s) \), given by

\[
L(s) = \frac{\delta^2 - \beta e + \beta s}{e-s}.
\]

If \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \in L^q(\mathbb{R}) \), then \( \Re L(s) < 0 \). Note that \( L \) maps \( e \) to \( \infty \). Therefore, given the line \( \Re s = 0 \), there exists a circle \( C \) passing through \( e \) such that \( L \) applies \( C \) into the line \( \Re s = 0 \). We claim that \( \Re \varepsilon > 0 \). In fact, assume that \( \Re \varepsilon < 0 \). Then \( \Re s < \Re \varepsilon < 0 \). Let \( s_0 \) be a point of the circle \( C \) satisfying \( \Re s_0 < 0 \) and \( \Re L(s_0) = 0 \). Assume that \( s \to s_0 \) with the restrictions \( \Re s \leq -e^* \) (\( e^* > 0 \)) and \( \Re L(s) < 0 \). Then \( g_s \) remains bounded in \( L^p(\mathbb{R}) \) as \( s \to s_0 \), while \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) blows up in \( L^q(\mathbb{R}) \). This is a contradiction because \( \mathcal{F}_{b_{, e, \delta, \xi, \gamma}} \) is bounded, and we conclude that \( \Re \varepsilon > 0 \).

In order to verify that \( \Re \beta > 0 \), let

\[
L_1(s) = L(s) + \beta = \frac{\delta^2}{e-s}.
\]
Taking $\text{Re} s = -\varepsilon^*$ $(\varepsilon^* > 0)$ and noting that $\text{Re} s < 0$ implies $\text{Re} L(s) < 0$, we have

(2.4) $\text{Re} L_1(s) = \text{Re} L(s) + \text{Re} \beta \leq \text{Re} \beta$.

Now, letting $s$ tend to infinity along the line $\text{Re} s = -\varepsilon^*$, we see that $0 \leq \lim_{s \to \infty} \text{Re} L_1(s) \leq \text{Re} \beta$, whence $\text{Re} \beta \geq 0$. Thus $\text{Re} \varepsilon \geq 0$ and $\text{Re} \beta \geq 0$.

Next suppose that $\text{Re} \varepsilon > 0$. In this case $L_1$ carries the line $\text{Re} s = 0$ into a circle $C_1$ passing through 0. By (2.4), to prove that $(\text{Re} \varepsilon) \cdot (\text{Re} \beta) \geq (\text{Re} \delta)^2$, it suffices to show that some point on that circle has real part

$$\frac{(\text{Re} \delta)^2 \cdot (\text{Re} \varepsilon)^{-1}}.$$

Denote such a point by $s_1$. From (2.4) we obtain

$$\text{Re} L_1(s_1) \leq \text{Re} \beta,$$

so that $(\text{Re} \varepsilon) \cdot (\text{Re} \beta) \geq (\text{Re} \delta)^2$. The center of the circle $C_1$ is $\frac{1}{2} L_1(s_2)$, where $s_2$ minimizes $|e - s|$ subject to the condition $\text{Re} s = 0$. It is easy to check that $\frac{1}{2} L_1(s_2) + |\frac{1}{2} L_1(s_2)|$ has the desired real part.

Finally, if $q < p$, we must show that the equality cannot hold in $(\text{Re} \varepsilon) \cdot (\text{Re} \beta) = (\text{Re} \delta)^2$. Indeed, if it did, then $\gamma^*$ could be chosen so that $\text{Re} \alpha^* = \text{Re} \beta^* = 0$ in (2.1). Then (2.1) would imply that $e^{(\gamma^*/4\delta)}$, with $\text{Re} (\gamma^*/4\delta) > 0$, is bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, which is false. □

3. Contraction properties

The purpose of this section is to give sufficient conditions in order that the operator $\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma}$ (for degenerate and nondegenerate cases), $\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{R}$, be a contraction over $L^p(\mathbb{R})$, $1 < p < \infty$, and a contraction as an operator from $L^2(\mathbb{R})$ into $L^p(\mathbb{R})$, $0 < p < \infty$.

The next results are motivated by Chapter 8 of [1].

Theorem 3.1. Let $1 < p < \infty$, and assume $\varepsilon > 0$ and $\delta^2 < p' \beta \varepsilon$ (here, $p'$ denotes the exponent conjugate to $p$). For all $f \in L^p(\mathbb{R})$, we have

(3.1) $\|\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma} f\|^p_p \leq H \cdot \int_{-\infty}^{+\infty} \exp \{[\delta^2/(\beta p - (p/p') \cdot (\delta^2/\varepsilon)) - \varepsilon]y^2 + \gamma + \delta \cdot (\xi p + (p/p') \cdot (\gamma \delta/\varepsilon))/(\beta p - (p/p') \cdot (\delta^2/\varepsilon))\} dy \cdot |f(y)|^p dy$,

where

$$H = \left(\frac{\pi}{\varepsilon}\right)^{p/2p'} \cdot \left(\frac{\pi}{(\beta p - (p/p') \cdot (\delta^2/\varepsilon))}\right)^{1/2} \cdot \exp\{(\gamma y^2/4p') + ((\xi p + (p/p') \cdot (\gamma \delta/\varepsilon))^2)/(4 \cdot (\beta p - (p/p') \cdot (\delta^2/\varepsilon)))\}.$$

Proof. By writing

$$\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma} f(x) = \int_{-\infty}^{+\infty} \{[\exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y]^{1/p} \cdot f(y)]
$$

$$+ \{[\exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y]^{1/p'} \cdot f(y)\} dy,$$
x ∈ ℝ, and applying Hölder's inequality, it follows that
\[
|\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)| \leq \left( \int_{-\infty}^{+\infty} \exp[-\beta x^2 - e y^2 + 2\delta xy + \xi x + \gamma y] \cdot |f(y)|^p \, dy \right)^{1/p}
\cdot \left( \int_{-\infty}^{+\infty} \exp[-\beta x^2 - e y^2 + 2\delta xy + \xi x + \gamma y] \, dy \right)^{1/p'}.
\]

Since for e > 0,
\[
\int_{-\infty}^{+\infty} \exp[-\beta x^2 - e y^2 + 2\delta xy + \xi x + \gamma y] \, dy = (\pi/e)^{1/2} \cdot \exp[((\xi^2/2e) - \beta)x^2 + \beta + (\xi \gamma/e)x + (\xi^2/4e)],
\]
we arrive at the estimate
\[
|\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)|^p \leq (\pi/e)^{p/2p'} \cdot \exp[p \gamma^2/4ep']
\cdot \int_{-\infty}^{+\infty} \exp[-(\beta p - (p/p') \cdot (\delta^2/e))x^2 - e y^2 + 2\delta xy
+ (\xi p + (p/p') \cdot (\gamma \delta/e))x + \gamma y] \cdot |f(y)|^p \, dy.
\]

After integration with respect to x, the theorem follows. □

**Corollary 3.1.** Under the same hypothesis and notation of Theorem 3.1, we set

\[ A = \delta^2/(\beta p - (p/p') \cdot (\delta^2/e)) - e \]

and

\[ B = \gamma + (\delta \cdot (\xi p + (p/p') \cdot (\gamma \delta/e)))/(\beta p - (p/p') \cdot (\delta^2/e)). \]

Then,

(a) If A = 0 (or equivalently, \( \delta^2 = \beta \cdot e \)) and \( B = 0 \), one has
\[
|\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)| \leq H \cdot f \quad \text{for all } f \in L^p(\mathbb{R}).
\]

Note that if \( H \leq 1 \), we obtain
\[
|\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)| \leq f \quad \text{for all } f \in L^p(\mathbb{R}).
\]

(b) If \( A < 0 \) (or equivalently, \( \delta^2 < \beta \cdot e \)), then
\[
|\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)| \leq (H \cdot \exp(B^2/4A))^{1/p} \cdot ||f||_p \quad \text{for all } f \in L^p(\mathbb{R}).
\]

Note that if \( H \cdot \exp(B^2/4A) \leq 1 \), we obtain
\[
|\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)| \leq f \quad \text{for all } f \in L^p(\mathbb{R}).
\]

**Remark.** For \( 1 < p < \infty \), \( e > 0 \), \( \delta \neq 0 \), and \( \delta^2 = \beta e \), the question of contractivity remains open if \( B \neq 0 \) or \( H > 1 \). This question also remains open for \( 1 < p < \infty \), \( e > 0 \), and \( \delta^2 < \beta e \), if \( H \cdot \exp(B^2/4A) > 1 \).

**Theorem 3.2.** Let \( 0 < p < \infty \), and assume \( e > 0 \) and \( \delta^2 < \beta e \). Then, for all \( f \in L^2(\mathbb{R}) \),
\[
||\mathcal{F}_{\beta, e, \delta, \xi, \gamma f}(x)||_p \leq H^* \cdot ||f||_2,
\]

(3.2)
where

\[ H^* = (2\pi)^{1/4} \cdot (4\varepsilon)^{-1/4} \cdot (\pi/p)^{1/2p} \cdot (\beta - (\delta^2/\varepsilon))^{-1/2p} \]
\[ \cdot \exp\left(\gamma^2/4\varepsilon + (p \cdot (\xi + (\delta y/\varepsilon))^2)/(4 \cdot (\beta - (\delta^2/\varepsilon)))\right). \]

**Proof.** By Schwarz’s inequality and the evaluation of a Gaussian integral, we obtain for \( f \in L^2(\mathbb{R}) \),

\[ |(\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma} f)(x)| \leq (2\pi)^{1/4} \cdot (4\varepsilon)^{-1/4} \]
\[ \cdot \exp\left[((\delta^2/\varepsilon) - \beta)x^2 + (\xi + (\delta y/\varepsilon))x + (\gamma^2/4\varepsilon)\right] \cdot \|f\|_2. \]

Again, by evaluating a Gaussian integral (3.2) follows. \( \square \)

**Corollary 3.2.** Under the same conditions and notation of Theorem 3.2, if \( H^* \leq 1 \), then

\[ \|\mathcal{F}_{\beta, \varepsilon, \delta, \xi, \gamma} f\|_p \leq \|f\|_2 \quad \text{for all } f \in L^2(\mathbb{R}). \]

**References**