FRAME PERTURBATIONS

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ABSTRACT. We consider the stability of Hilbert space frames under perturbations. Our results are in spirit close to classical results for orthonormal bases, due to Paley and Wiener.

A frame can be viewed as a "generalized orthonormal basis"; if \( \{f_i\}_{i \in I} \) is a frame for the Hilbert space \( \mathcal{H} \), then any \( f \in \mathcal{H} \) can be written as an infinite linear combination of the elements \( f_i \). The coefficients do not need to be unique, and in general the expansion is nonorthogonal. But frames are a much more flexible tool than orthonormal bases, and they play a big role in wavelet theory.

It is a classical result that a sufficiently small perturbation of an orthonormal basis gives a Riesz basis. Our aim here is to consider the similar problem for frames. Our approach is motivated by the book [Y] and a result in [H] about perturbations of atoms in Banach spaces.

Let \( \mathcal{H} \) be a separable Hilbert space, with the inner product \( \langle \cdot, \cdot \rangle \) linear in the first entry.

A family \( \{f_i\}_{i \in I} \) of elements in \( \mathcal{H} \) is called a Bessel sequence if

\[
\exists B > 0: \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2 \quad \forall f \in \mathcal{H}.
\]

If \( \{f_i\}_{i \in I} \) is a Bessel sequence, then \( \sum_{i \in I} c_i f_i \) converges unconditionally for all \( \{c_i\} \in l^2(I) \) and the mapping \( T: \{c_i\} \mapsto \sum_{i \in I} c_i f_i \) is bounded from \( l^2(I) \) into \( \mathcal{H} \), with \( \|T\| \leq \sqrt{B} \). Composing \( T \) with the adjoint operator \( T^* : f \mapsto \{\langle f, f_i \rangle\}_{i \in I} \) we get the frame operator

\[
S: \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i \in I} \langle f, f_i \rangle f_i.
\]

The Bessel sequence \( \{f_i\}_{i \in I} \) is called a frame if

\[
\exists A > 0: A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \quad \forall f \in \mathcal{H}.
\]
Any pair of numbers \( A \) and \( B \) such that (1) resp. (2) are satisfied will be called a set of frame bounds. The smallest possible upper bound is

\[
\sup_{\|f\|=1} \sum_{i \in I} |\langle f, f_i \rangle|^2 = \sup_{\|f\|=1} |\langle Sf, f \rangle| = \|S\| = \|T\|^2.
\]

If \( \{f_i\}_{i \in I} \) is a frame, then \( S \) has a bounded inverse, defined on all of \( \mathcal{H} \); this fact leads to the important frame decomposition

\[
f = SS^{-1}f = \sum_{i \in I} \langle S^{-1}f, f_i \rangle f_i = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i \quad \forall f \in \mathcal{H}.
\]

\( \{S^{-1}f_i\}_{i \in I} \) is also a frame, usually called the dual frame; as bounds one can use \( \frac{1}{B} \) and \( \frac{1}{A} \).

**Theorem 1.** Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \), with bounds \( A \) and \( B \). Any family \( \{g_i\}_{i \in I} \) of elements in \( \mathcal{H} \) such that

\[
R := \sum_{i \in I} \|f_i - g_i\|^2 < A
\]

is a frame for \( \mathcal{H} \) with bounds \( A(1 - \sqrt{\frac{R}{A}})^2 \) and \( B(1 + \sqrt{\frac{R}{B}})^2 \).

**Proof.** Denote the frame operator for \( \{f_i\}_{i \in I} \) by \( S \). The assumptions imply that \( \{g_i\}_{i \in I} \) is a Bessel sequence, so we can define a bounded linear operator

\[
U: \mathcal{H} \to \mathcal{H}, \quad Uf := \sum_{i \in I} \langle f, S^{-1}f_i \rangle g_i.
\]

Now,

\[
\|f - Uf\|^2 = \left\| \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i - \langle f, S^{-1}f_i \rangle g_i \right\|^2 \\
\leq \sum_{i \in I} |\langle f, S^{-1}f_i \rangle|^2 \cdot \sum_{i \in I} \|f_i - g_i\|^2 \\
\leq \frac{R}{A} \|f\|^2 \quad \forall f \in \mathcal{H}.
\]

That is, \( \|I - U\| \leq \sqrt{\frac{R}{A}} < 1 \). So \( U \) is invertible, and

\[
\|U\| \leq 1 + \sqrt{\frac{R}{A}}, \quad \|U^{-1}\| \leq \frac{1}{1 - \sqrt{\frac{R}{A}}}
\]

Any \( f \in \mathcal{H} \) can be written as

\[
f = UU^{-1}f = \sum_{i \in I} \langle U^{-1}f, S^{-1}f_i \rangle g_i;
\]

thus

\[
\|f\|^4 = \left\| \sum_{i \in I} \langle U^{-1}f, S^{-1}f_i \rangle g_i, f \right\|^2 = \left| \sum_{i \in I} \langle U^{-1}f, S^{-1}f_i \rangle \langle g_i, f \rangle \right|^2 \\
\leq \sum_{i \in I} |\langle U^{-1}f, S^{-1}f_i \rangle|^2 \cdot \sum_{i \in I} |\langle g_i, f \rangle|^2 \\
\leq \frac{1}{A} \|U^{-1}f\|^2 \sum_{i \in I} |\langle g_i, f \rangle|^2 \leq \frac{\|f\|^2}{A(1 - \sqrt{\frac{R}{A}})^2} \cdot \sum_{i \in I} |\langle g_i, f \rangle|^2 \quad \forall f \in \mathcal{H}.
\]
So
\[ A \left(1 - \sqrt{\frac{R}{A}}\right)^2 \|f\|^2 \leq \sum_{i \in I} |(g_i, f)|^2 \quad \forall f \in \mathcal{H}. \]

Now define
\[ T: l^2(I) \to \mathcal{H}, \quad T\{c_i\} := \sum_{i \in I} c_i g_i. \]

The frame operator for \( \{g_i\}_{i \in I} \) is \( TT^* \), so the optimal upper frame bound for \( \{g_i\}_{i \in I} \) is \( \|T\|^2 \). For \( \{c_i\} \in l^2(I) \) we have
\[ \|T\{c_i\}\| = \left\| \sum_{i \in I} c_i g_i \right\| \leq \left\| \sum_{i \in I} c_i (g_i - f_i) \right\| + \left\| \sum_{i \in I} c_i f_i \right\| \leq (\sqrt{B} + \sqrt{R}) \|\{c_i\}\|. \]

So
\[ \|T\|^2 \leq (\sqrt{B} + \sqrt{R})^2 = B \left(1 + \sqrt{\frac{R}{B}}\right)^2. \]

In some sense, the result is the best possible; if \( \sum_{i \in I} \|f_i - g_i\|^2 = A \), then \( \{g_i\}_{i \in I} \) does not even need to be total in \( \mathcal{H} \). For example, if \( \{f_i\}_{i=1}^\infty \) is an ONB, then \( \{f_i\}_{i=1}^\infty \) is a frame with \( A = B = 1 \). If we define \( g_1 = 0, \ g_i = f_i, i \geq 2 \), then \( \sum_{i \in I} \|f_i - g_i\|^2 = 1 \) and \( \{g_i\} \) is not total.

**Lemma 2.** Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \). If \( J \subseteq I \) is finite, then \( \{f_i\}_{i \in I-J} \) is a frame for \( \text{span}\{f_i\}_{i \in I-J} \).

**Proof.** Let \( j \in I \); it is enough to prove that \( \{f_i\}_{i \neq j} \) is a frame for \( \text{span}\{f_i\}_{i \neq j} \). Let \( P \) denote the orthogonal projection on \( \text{span}\{f_i\}_{i \neq j} \). Then \( \{f_i\}_{i \neq j} \cup \{Pf_i\} \) is a frame for \( \text{span}\{f_i\}_{i \neq j} \). But \( \{f_i\}_{i \neq j} \) is total in \( \text{span}\{f_i\}_{i \neq j} \) and therefore itself a frame for \( \text{span}\{f_i\}_{i \neq j} \); cf. [DS, Lemma 9].

Two families \( \{f_i\}_{i \in I} \) and \( \{g_i\}_{i \in I} \) are said to be quadratically close if
\[ \sum_{i \in I} \|f_i - g_i\|^2 < \infty. \]

**Theorem 3.** Let \( \{f_i\}_{i \in I} \) be a frame and \( \{g_i\}_{i \in I} \) a family which is quadratically close to \( \{f_i\}_{i \in I} \). Then \( \{g_i\}_{i \in I} \) is a frame for \( \text{span}\{g_i\}_{i \in I} \).

**Proof.** Again let \( A \) denote a lower frame bound for \( \{f_i\}_{i \in I} \). There exists a finite index set \( J \subseteq I \) such that \( \sum_{i \in I-J} \|f_i - g_i\|^2 < A \). By Theorem 1 \( \{f_i\}_{i \in J} \cup \{g_i\}_{i \in I-J} \) is a frame for \( \mathcal{H} \). Now Lemma 2 shows that \( \{g_i\}_{i \in I-J} \) is a frame for \( \text{span}\{g_i\}_{i \in I-J} \).

\( \{g_i\}_{i \in I} \) is a Bessel sequence in \( \mathcal{H} \). Observe that
\[ \text{span}\{g_i\}_{i \in I} = \text{span}\{g_i\}_{i \in I-J} + \text{span}\{g_i\}_{i \in J}; \]

it follows that the operator
\[ T: l^2(I) \to \text{span}\{g_i\}_{i \in I}, \quad T\{c_i\} := \sum_{i \in I} c_i g_i \]

is surjective. Now [C, Corollary 4.2] implies that \( \{g_i\}_{i \in I} \) is a frame for \( \text{span}\{g_i\}_{i \in I} \).

**Remark.** Let \( G \) be a topological group and \( \pi \) a strongly continuous unitary representation of \( G \) on the Hilbert space \( \mathcal{H} \). In wavelet analysis one is interested
in coherent frames, i.e., frames of the form \( \{ \pi(x_i) \}_{i \in I} \), where \( \{ x_i \}_{i \in I} \) is a set of group elements and \( f \in \mathcal{H} \) (see [D, DGM, HW]). Our results cannot be applied to perturbations of \( f \), since \( \| \pi(x_i)f - \pi(x_i)g \| = \| f - g \| \). But if \( \{ y_i \}_{i \in I} \) is another family of group elements, then \( \| \pi(x_i)f - \pi(y_i)f \| = \| f - \pi(x_i^{-1}y_i)f \| \). So our results show that \( \{ \pi(y_i)f \}_{i \in I} \) is a frame if \( \{ y_i \}_{i \in I} \) is sufficiently close to \( \{ x_i \}_{i \in I} \). We shall not go into details with concrete calculations here.

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REFERENCES


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