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# ON AN IDENTITY RELATED TO MULTIVALENT FUNCTIONS

A. Z. GRINSHPAN

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*Dedicated to Professor A. W. Goodman on his 80th birthday*

ABSTRACT. We prove an algebraic identity by induction. This identity is very important in the coefficient problem for analytic functions that are $p$-valent in the unit disk.

## 1. INTRODUCTION

In 1948 Goodman [1] proposed the conjecture that if

$$f(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

is regular and $p$-valent in $E\colon |z| < 1$, then the coefficients satisfy the inequality

$$|b_n| \le \sum_{k=1}^{p} D(p, k, n)|b_k|, \qquad 1 \le k \le p < n, \tag{2}$$

where by definition

$$D(p, k, n) = \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}. \tag{3}$$

It was proved in [1] that if the conjecture were true it would be sharp for every selection of the $p$ variables $|b_1|, |b_2|, \ldots, |b_p|$. In a very nice paper Goodman and Robertson [2] proved that the conjecture (2) is true for the class $T(p)$, a very large class of multivalent functions. The class $T(p)$ is the natural extension of the Rogosinski class of typically-real functions to the class of typically-real functions of order $p$. Further, $T(p)$ includes the class of all $p$-valent starlike functions with real coefficients.

The proof in [2] depends on an algebraic identity symbolized by

$$D^*(p, k, n) = D(p, k, n), \qquad 1 \le k \le p < n, \tag{4}$$

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where $D^*(p, k, n)$ will be defined later. The proof of (4) given in [2] is quite sophisticated and subtle, but can also be regarded as artificial and unsatisfactory because it is not direct. In this paper we give a direct proof of (4).

## 2. The algebraic identity

In [2], equation (3.17) in that paper, the quantities $D^*(p, k, n)$ are defined by equating the corresponding coefficients of $|c_k^{(p)}|$ on both sides of the equation

(3.17)
$$\sum_{k=1}^{p} D^*(p, k, n)|c_k^{(p)}| = (n-p)|c_{p-1}^{(p)}| + (n-p+1)|c_p^{(p)}| + \sum_{j=p}^{n-1}(n-j)\sum_{s=1}^{p-1} D(p-1, s, j)[|c_{s+1}^{(p)}| + 2|c_s^{(p)}| + |c_{s-1}^{(p)}|],$$

where we set $c_0^{(p)} = 0$.

In this equation the absolute value signs are not needed (for our purposes) and the superscripts in $c_k^{(p)}$ are unnecessary. Henceforth we drop both of these items.

**Theorem.** *Equation* (4) *holds for every set of positive integers satisfying* $1 \le k \le p < n$.

## 3. The proof

We first regroup the terms in (3.17) by changing the range of the indices. The standard technique will give

$$\begin{aligned}\sum_{k=1}^{p} D^*(p, k, n)c_k &= (n-p)c_{p-1} + (n-p+1)c_p \\ &+ \sum_{j=p}^{n-1}(n-j)\sum_{s=2}^{p} D(p-1, s-1, j)c_s \\ &+ \sum_{j=p}^{n-1}(n-j)\sum_{s=1}^{p-1} D(p-1, s, j)2c_s \\ &+ \sum_{j=p}^{n-1}(n-j)\sum_{s=0}^{p-2} D(p-1, s+1, j)c_s.\end{aligned} \tag{5}$$

Equating coefficients in (5), will give $p$ equations but special attention must be given when $k = p$ or $k = p-1$. Further, when $p \ge 4$, one must consider three different cases. To avoid this difficulty we extend the definition of $D(p, k, n)$ in a natural way. Let

$$D(p, 0, j) = 0 \qquad \text{all } p, j, \tag{6}$$

$$D(p-1, p, j) = \begin{cases} 1, & \text{if } j = p, \\ 0, & \text{if } j > p, \end{cases} \tag{7}$$

$$D(p-1, p+1, j) = \begin{cases} 0, & \text{if } j = p, \\ -1, & \text{if } j = p+1, \\ 0, & \text{if } j > p+1. \end{cases} \tag{8}$$

Then (5), (6), (7) and (8) give

$$D^*(p, k, n) = \sum_{j=p}^{n-1} (n-j)[D(p-1, k-1, j) + 2D(p-1, k, j) + D(p-1, k+1, j)] \tag{9}$$

for $1 \le k \le p < n$.

We use induction on $n$ to prove that (4) holds for all $n = p+1, p+2, \dots$. For $n = p+1$, equation (9) gives

$$\begin{aligned} D^*(p, k, p+1) &= D(p-1, k-1, p) + 2D(p-1, k, p) + D(p-1, k+1, p) \\ &= \frac{2(2p-1)!}{(p+k)!(p-k)!}\left[\frac{(k-1)(p+k)}{(p-k+1)} + 2k + \frac{(k+1)(p-k)}{(p+k+1)}\right]. \end{aligned} \tag{10}$$

The sum inside the brackets will give $(4kp^2+2kp)/[(p+1)^2-k^2]$, after a small computation. Hence $D^*(p, k, n) = D(p, k, n)$ when $n = p+1$.

Now assume that (4) is true for $n = p+1, p+2, \dots, N$. Thus

$$D(p, k, n) = \sum_{j=p}^{n-1} (n-j)[D(p-1, k-1, j) + 2D(p-1, k, j) + D(p-1, k+1, j)] \tag{11}$$

for $n = p+1, p+2, \dots, N$. We must examine

$$D^*(p, k, N+1) = \sum_{j=p}^{N} (N+1-j)[D(p-1, k-1, j) + 2D(p-1, k, j) + D(p-1, k+1, j)].$$

We break the first factor into two parts, 1 and $N-j$, and observe that in the second sum we have 0 when $j = N$. Consequently

$$\begin{aligned} D^*(p, k, N+1) &= \sum_{j=p}^{N} [D(p-1, k-1, j) + 2D(p-1, k, j) + D(p-1, k+1, j)] \\ &\quad + \sum_{j=p}^{N-1} (N-j)[D(p-1, k-1, j) + 2D(p-1, k, j) + D(p-1, k+1, j)]. \end{aligned}$$

By the induction hypothesis (11), the second sum is $D(p, k, N)$, so

$$\begin{aligned} D^*(p, k, N+1) &= D(p, k, N) + \sum_{j=p}^{N} [D(p-1, k-1, j) + 2D(p-1, k, j) \\ &\quad + D(p-1, k+1, j)]. \end{aligned} \tag{12}$$

To complete the proof, we do a second induction. In (12) we replace $N$ by $n-1$ and define $D^{**}(p,k,n)$ by

$$D^{**}(p,k,n) = D(p,k,n-1) + \sum_{j=p}^{n-1}[D(p-1,k-1,j) + 2D(p-1,k,j) + D(p-1,k+1,j)]. \tag{13}$$

We will use induction to prove that $D^{**}(p,k,n) = D(p,k,n)$ for $n \geq p+2$ and hence conclude that $D^{*}(p,k,n) = D(p,k,n)$. We observe that with the natural extension $D(p,p,p) = 1$. Further if $k=p$ and $n=p+1$ then (10) and (13) give $D^{**}(p,p,p+1) = 1 + D(p,p,p+1) \neq D(p,p,p+1)$.

When $n = p+2$, equation (13) gives

$$\begin{aligned} D^{**}(p,k,p+2) &= D(p,k,p+1) \\ &\quad + D(p-1,k-1,p) + 2D(p-1,k,p) + D(p-1,k+1,p) \\ &\quad + D(p-1,k-1,p+1) + 2D(p-1,k,p+1) \\ &\quad + D(p-1,k+1,p+1). \end{aligned} \tag{14}$$

From (10) the second line in (14) gives $D(p,k,p+1)$. Hence

$$\begin{aligned} D^{**}(p,k,p+2) = 2D(p,k,p+1) + D(p-1,k-1,p+1) \\ + 2D(p-1,k,p+1) + D(p-1,k+1,p+1). \end{aligned} \tag{15}$$

Using the definition of $D(p,k,n)$ and dropping $0! = 1! = 1$, we have

$$\begin{aligned} D^{**}(p,k,p+2) = 2&\frac{2k(2p+1)!}{(p+k)!(p-k)![(p+1)^2-k^2]} \\ &+ \frac{2(k-1)(2p)!}{(p+k-2)!(p-k)![(p+1)^2-(k-1)^2]} \\ &+ 2\frac{2k(2p)!(p-k)}{(p+k-1)!(p-k)![(p+1)^2-k^2]} \\ &+ \frac{2(k+1)(2p)!(p-k)(p-k-1)}{(p+k)!(p-k)![(p+1)^2-(k+1)^2]}, \end{aligned}$$

or

$$D^{**}(p,k,p+2) = \frac{2(2p)!}{(p+k)!(p-k)!}[s_1+s_2+s_3+s_4]. \tag{16}$$

Here by definition,

$$s_1 + s_3 = \frac{2k(2p+1)}{(p+1)^2-k^2} + \frac{2k(p^2-k^2)}{(p+1)^2-k^2} = 2k, \tag{17}$$

and

$$\begin{aligned} s_2 + s_4 &= \frac{(k-1)(p+k-1)}{p-k+2} + \frac{(k+1)(p-k-1)}{p+k+2} \\ &= \frac{2k(p^2-p+k^2-3)}{(p+2)^2-k^2}. \end{aligned} \tag{18}$$

Thus $s_1+s_2+s_3+s_4=k(2p+1)(2p+2)/[(p+2)^2-k^2]$, and hence from (16)

$$D^{**}(p,k,p+2)=\frac{2k(p+2)!}{(p+k)!(p-k)![(p+2)^2-k^2]}=D(p,k,p+2). \tag{19}$$

We now apply mathematical induction to the statement that $D^{**}(p,k,n)=D(p,k,n)$ assuming that it is true for indices $n=p+2, p+3, \ldots, L$. From (13) we have

$$\begin{aligned} &D^{**}(p,k,L+1)\\ &\quad=D(p,k,L)+\sum_{j=p}^{L}[D(p-1,k-1,j)\\ &\qquad\qquad+2D(p-1,k,j)+D(p-1,k+1,j)] \end{aligned} \tag{20}$$

or

$$\begin{aligned} D^{**}(p,k,L+1)=D(p,k,L)-D(p,k,L-1)+D(p,k,L-1)\\ +\sum_{j=p}^{L}[D(p-1,k-1,j)+2D(p-1,k,j)\\ +D(p-1,k+1,j)]. \end{aligned} \tag{21}$$

But by the induction hypothesis

$$\begin{aligned} D(p,k,L-1)+\sum_{j=p}^{L-1}[D(p-1,k-1,j)+2D(p-1,k,j)\\ +D(p-1,k+1,j)]\\ =D(p,k,L). \end{aligned}$$

So (21) becomes

$$\begin{aligned} D^{**}(p,k,L+1)&=2D(p,k,L)-D(p,k,L-1)+D(p-1,k-1,L)\\ &\quad+2D(p-1,k,L)+D(p-1,k+1,L)\\ &=2\frac{2k(L+p)!}{(p+k)!(p-k)!(L-p-1)!(L^2-k^2)}\\ &\quad-\frac{2k(L+p-1)!}{(p+k)!(p-k)!(L-p-2)![(L-1)^2-k^2]}\\ &\quad+\frac{2(k-1)(L+p-1)!}{(p+k-2)!(p-k)!(L-p)![L^2-(k-1)^2]}\\ &\quad+2\frac{2k(L+p-1)!(p-k)}{(p+k-1)!(p-k)!(L-p)!(L^2-k^2)}\\ &\quad+\frac{2(k+1)(L+p-1)!(p-k)(p-k-1)}{(p+k)!(p-k)!(L-p)![L^2-(k+1)^2]}. \end{aligned} \tag{22}$$

The first and fourth terms in the above expression combine to give

$$(2k)\frac{2(L+p-1)!}{(p+k)!(p-k)!(L-p)!}.$$

Then $D^{**}(p,k,L+1)$ can be written as

$$D^{**}(p,k,L+1)=\frac{2(L+p-1)!}{(p+k)!(p-k)!(L-p)!}I(p,k,L), \tag{23}$$

where

$$I(p\,,\,k\,,\,L)=2k-\frac{k(L-p)(L-p-1)}{(L-1)^2-k^2}+\frac{(k-1)(p+k)(p+k-1)}{L^2-(k-1)^2}+\frac{(k+1)(p-k)(p-k-1)}{L^2-(k+1)^2}. \tag{24}$$

To simplify $I(p\,,\,k\,,\,L)$ we use one $k$ and the next two terms in (24) and set

$$\Phi(p\,,\,k\,,\,L)=k-\frac{k(L-p)(L-p-1)}{(L-1)^2-k^2}+\frac{(k-1)(p+k)(p+k-1)}{L^2-(k-1)^2}. \tag{25}$$

Observing that the L.C.D. in (25) is $[(L-k)^2-1](L+k-1)$, we have

$$\begin{aligned}&\Phi(p\,,\,k\,,\,L)\\&\quad=k+\frac{1}{L+k-1}\\&\qquad\cdot\left(\frac{-k(L-p)(L-p-1)(L+1-k)+(k-1)(p+k)(p+k-1)(L-1-k)}{(L-k)^2-1}\right).\end{aligned}$$

Let $\mathscr{N}$ be the numerator of the last term in $\Phi$. We write $\mathscr{N}$ as a polynomial in $p$ and after a moderate computation we find that

$$\begin{aligned}\mathscr{N}&=p^2(-L-k+1)+p(L+k-1)(2Lk-2k^2+1)\\&\quad+k(L+k-1)(-L^2+2kL-L+1-k^2).\end{aligned} \tag{26}$$

After an obvious cancellation, this gives

$$\begin{aligned}\Phi(p\,,\,k\,,\,L)&=k+\frac{-p^2+(2Lk-2k^2+1)p+k(-L^2+2kL-L+1-k^2)}{(L-k)^2-1}\\&=\frac{-p^2+(2Lk-2k^2+1)p-kL}{(L-k)^2-1}.\end{aligned} \tag{27}$$

From this last expression it is clear that $\Phi(p\,,\,k\,,\,L)=\Phi(p\,,\,-k\,,\,-L)$ and using this in (25) we find that

$$\Phi(p\,,\,-k\,,\,-L)=-k+\frac{k(L+p)(L+p+1)}{(L+1)^2-k^2}-\frac{(k+1)(p-k)(p-k-1)}{L^2-(k+1)^2}. \tag{28}$$

From (24) and (25) $I(p\,,\,k\,,\,L)$ can be put in the form

$$I(p\,,\,k\,,\,L)=k+\Phi(p\,,\,k\,,\,L)+\frac{(k+1)(p-k)(p-k-1)}{L^2-(k+1)^2}. \tag{29}$$

When $\Phi(p\,,\,k\,,\,L)$ is replaced by $\Phi(p\,,\,-k\,,\,-L)$ given by (28) we find that

$$I(p\,,\,k\,,\,L)=\frac{k(L+p)(L+p+1)}{(L+1)^2-k^2}.$$

Combining this with (23) we have

$$\begin{aligned}D^{**}(p\,,\,k\,,\,L+1)&=\frac{2(L+p-1)!}{(p+k)!(p-k)!(L-p)!}\cdot\frac{k(L+p)(L+p+1)}{(L+1)^2-k^2}\\&=D(p\,,\,k\,,\,L+1).\end{aligned}$$

## References

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Department of Mathematics, University of South Florida, Tampa, Florida 33620
*E-mail address*: azg@math.usf.edu

Russian Academy of Sciences, INENCO, St. Petersburg, 190000, Russia