EXISTENCE OF AN ANGULAR DERIVATIVE
FOR A CLASS OF STRIP DOMAINS

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Abstract. A strip domain $R$ is said to have an angular derivative if for each
conformal map $\phi: R \to S = \{z : |\text{Im } z| < 1/2\}$ the limit $\lim_{\text{Re } w \to +\infty} (\phi(w) - w)$ exists
and is finite as $\text{Re } w \to +\infty$. Rodin and Warschawski considered a class of strip
domains for which the euclidean area of $S \setminus R'$ is finite, where $R'$ denotes
a Lipschitz approximation of $R$, $R' \subset R$. They showed that a sufficient
condition for an angular derivative to exist is that the euclidean area of $R' \setminus S$
be finite. We prove that this condition is also necessary.

Introduction

Let $R$ be a simply-connected region in the plane, containing the real axis and
such that $\partial R$ meets both half-planes. Thus $R$ contains $\pm \infty$ as prime ends.
Let $\phi: R \to S$ be a one-to-one conformal map of $R$ to $S = \{z : |\text{Im } z| < 1/2\}$ such that $\pm \infty$ correspond. The mapping $\phi$ has an angular derivative $c$
$(-\infty < c < +\infty)$ at $+\infty$ if the following two conditions are satisfied:

1. For each $\delta > 0$ there is a $U_\delta$ such that
   \[ R_\delta = \{w : \text{Re } w > U_\delta, -1/2 + \delta < \text{Im } w < 1/2 - \delta\} \subset R, \]

2. For each $\delta$ satisfying $0 < \delta < 1/2,$ \( \lim_{\text{Re } w \to +\infty} (\phi(w) - w) = c. \)

If one such map $\phi$ has an angular derivative at $+\infty$, then so do all the
others and we say that $R$ has an angular derivative. The “problem of angular
derivative” is to find euclidean geometric conditions on $R$ such that an angular
derivative exists. A moment’s reflection shows that the existence or otherwise
of the angular derivative pertains to how closely the given strip $R$ matches the
standard strip $S$ near $+\infty$ in a conformal sense.

Let $S^\lambda_\mu$ denote the euclidean strip bounded by \{x + iy : y = \lambda\} above and
\{x + iy : y = -\mu\} below for $\lambda$ and $\mu$ positive. Given a strip $R$ of the type
specified above, let $\phi^\lambda_\mu: R \to S^\lambda_\mu$ be a conformal map so that $\pm \infty$ correspond.

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the National Science Foundation.
It is then trivial to check that the angular derivative at $+\infty$ can exist for $\phi_\lambda^\mu$ only for at most one choice of $\lambda$, $\mu$. By taking a suitable linear transformation of both $R$ and $S^\mu_\lambda$ we normalize to the situation $\lambda = \mu = 1/2$.

The following definition of Lipshitz-1 minorants of $\partial R$ is as in [6]. Consider Lipshitz-1 functions $f : \mathbb{R} \to \mathbb{R}$. Let $B^+$ be the family of these $f$ for which $\partial R \cap \{w : \text{Im } w > 0, \text{Re } w > 0\}$ lies above the graph $\{(u, f(u) + 1/2) : u \in \mathbb{R}\}$. Let $B^-$ be the family of these $f$ for which $\partial R \cap \{w : \text{Im } w < 0, \text{Re } w > 0\}$ lies below the graph $\{(u, f(u) - 1/2) : u \in \mathbb{R}\}$. Define $h_\pm : \mathbb{R} \to \mathbb{R}$ by

$$ h_+(u) = \sup_{f \in B^+} f(u), \quad h_-(u) = \inf_{f \in B^-} f(u). $$

Then $h_+$ and $h_-$ are the Lipshitz-1 approximations to $\partial R$, from inside $R$. We prove:

**Theorem 1.** Let $R$ be as above, and assume that

$$ \int_0^\infty \min(h_+(u), 0) \, du > -\infty \quad \text{and} \quad \int_0^\infty \max(h_-(u), 0) \, du < \infty. $$

Then $R$ has an angular derivative at $\infty$ if and only if

$$ \int_0^\infty \max(h_+(u), 0) \, du < \infty \quad \text{and} \quad \int_0^\infty \min(h_-(u), 0) \, du > -\infty. $$

A sketch quickly demonstrates that conditions (3) and (4) have simple geometrical meaning. In fact condition (3) states that certain “inner areas”, relative to $S$, are assumed to be finite, and condition (4) then asserts that for the existence of an angular derivative the corresponding “outer areas” must also be finite. Clearly this ties in with how well the standard strip $S$ approximates the Lipshitz modification of $R$.

Rodin and Warschawski formulated the above statement in [6] and proved that (4) is sufficient for an angular derivative to exist for a strip region $R$ which satisfies (3). We prove that condition (4) is also necessary when (3) holds.

**Remark.** A related result is that of Burdzy [2, Theorem 7.1] which is stated in the half-plane setting and proved using probabilistic methods. Later Carroll [3] and then Gardiner [4] gave complex analysis proofs of Burdzy’s Theorem. Rodin and Warschawski [6] claim that Theorem 1 is equivalent to Burdzy’s Theorem. However the author fails to see a rigorous proof of that equivalence. One difficulty is that it is not clear that Lipschitz minorants in the half-plane and strip regions correspond.

The situation when condition (3) fails is still open.

**Proof.** As indicated above, we only show the necessity of (4). Suppose $R$ has an angular derivative at $+\infty$. Let $\theta_u$ denote that cross-cut of $R$ which intersects the real axis and which lies on the vertical line having real part $u$. For $u_1 < u_2$, let $\lambda_R(u_1, u_2)$ be the extremal length of all arcs in $R$ which join the cross-cuts $\theta_{u_1}$ to $\theta_{u_2}$ and lie in the component of $R - \theta_{u_1} - \theta_{u_2}$ which contains each $\theta_u$, $u_1 < u < u_2$. Since $R$ has an angular derivative, it follows from [7, Theorem 6] that

$$ \lambda_R(u_1, u_2) = u_2 - u_1 + o(1), $$

where $o(1) \to 0$ as $u_2 > u_1 \to +\infty$. 
Let \( R' \) be the right half-plane portion of the strip region bounded by (and not including)
\[
\beta_+ = \{(u, h_+(u) + 1/2) \mid u \in \mathbb{R}\}, \quad \beta_- = \{(u, h_-(u) - 1/2) \mid u \in \mathbb{R}\}.
\]
Note that \( R' \subset R \), and by the comparison principle [1, Theorem 4.1], we have \( \lambda_R(u_1, u_2) \leq \lambda_{R'}(u_1, u_2) \). So that by (5),
\[
\lambda_{R'}(u_1, u_2) \geq u_2 - u_1 + o(1). \tag{6}
\]
We now obtain an upper bound for \( \lambda_{R'}(u_1, u_2) \). To do this, consider the conjugate extremal distance \( 1/\lambda_{R'}(u_1, u_2) \) [1, p. 53]. We will introduce a specific metric \( \rho \) to obtain a lower bound for \( 1/\lambda_{R'}(u_1, u_2) \).

**Lemma 1.** If condition (3) holds and \( R \) has an angular derivative at \( \infty \), then \( h_\pm(u) \to 0 \) as \( u \to +\infty \).

*Proof.* That \( \min(h_+(u), 0) \to 0 \) and \( \max(h_-(u), 0) \to 0 \) can be seen easily from condition (1) in the definition of angular derivative and since \( h_+ \) and \( h_- \) are Lipschitz minorants to \( dR \).

Further, \( \max(h_+(u), 0) \to 0 \) and \( \min(h_-(u), 0) \to 0 \) follow directly from [5, p. 102, Proposition 1], since \( R' \subset R \). However for the sake of completeness we include a proof.

Suppose, by contradiction, that \( \max(h_+(u), 0) \to 0 \). Hence there exist \( \epsilon_0 > 0 \) and \( \{u_i\}, u_i \to +\infty \) such that \( h_+(u_i) > \epsilon_0 \) for all \( i \). Set \( s_i = u_i - \epsilon_0/2, t_i = u_i + \epsilon_0/2 \). Now \( \max(h_-(u), 0) \to 0 \) implies that there exists \( U \) such that \( \max(h_-(u), 0) < \epsilon_0/4 \) for all \( u \geq U \). Since \( h_+ \) is Lipschitz-1, the comparison principle [1, Theorem 4.1] shows that for all \( i \) such that \( u_i \geq U + \epsilon_0 \), we have \( \lambda_{R'}(s_i, t_i) \leq (t_i - s_i)/(1 + \epsilon_0/2 - \epsilon_0/4) \) and (6) yields that \( (t_i - s_i)/(1 + \epsilon_0/4) \geq t_i - s_i + o(1) \), as \( i \to \infty \). But , \( t_i - s_i \equiv \epsilon_0 \) and the above is a contradiction. Hence \( h_+(u) \to 0 \) as \( u \to +\infty \). Similarly we can show that \( h_-(u) \to 0 \).

With no loss of generality we assume \( |h_\pm(u)| \leq 1/8 \) for \( u \geq 0 \). To enable us to define \( \rho \), we set for \( u \in \mathbb{R} \)
\[
h_+^*(u) = \begin{cases} 2h_+(u), & \text{if } h_+(u) < 0, \\ -\frac{1}{2}h_+(u), & \text{if } h_+(u) \geq 0 
\end{cases}
\]
and
\[
h_-^*(u) = \begin{cases} 2h_-(u), & \text{if } h_-(u) > 0, \\ -\frac{1}{2}h_-(u), & \text{if } h_-(u) \leq 0.
\end{cases}
\]
Let \( R^* \) be the right half-plane portion of the strip region bounded by (and not including)
\[
\beta_+^* = \{(u, h_+^*(u) + 1/2) \mid u \in \mathbb{R}\}, \quad \beta_-^* = \{(u, h_-^*(u) - 1/2) \mid u \in \mathbb{R}\}.
\]
Note that \( R^* \subset R' \cap S \). Let \( 0 < u_1 < u_2 \) and \( Q = Q(u_1, u_2) = R' \cap \{u + iv : u_1 < u < u_2\} \)
\[
R_1 = \{u + iv \in Q : h_+(u) > 0, v > 0\}, \quad R_2 = \{u + iv \in Q : h_-(u) < 0, v < 0\}, \\
R_3 = \{u + iv \in Q : h_+(u) < 0, v > 0\}, \quad R_4 = \{u + iv \in Q : h_-(u) > 0, v < 0\}.
\]
Define a metric $\rho$ on $Q(u_1, u_2)$:

$$
\rho(u + iv) = \begin{cases} 
1, & u + iv \in \mathbb{R}^* \cap Q, \\
1/2, & u + iv \in (R_1 \cup R_2) - \mathbb{R}^*, \\
2\sqrt{2}, & u + iv \in (R_3 \cup R_4) - \mathbb{R}^*.
\end{cases}
$$

(7)

Clearly $\rho$ is well defined and Borel measurable. Let $\Gamma = \Gamma(u_1, u_2)$ be the family of all connected, rectifiable arcs which have one end-point on $\beta_+$, one end-point on $\beta_-$, and all other points in $Q$. By [1, 4-1, 4-2], it is evident that

$$
\frac{1}{\lambda_R(u_1, u_2)} = \lambda_\rho(\Gamma).
$$

(8)

Lemma 2. If $\rho$ is as above, then

$$
L(\Gamma, \rho) \equiv \inf_{\gamma \in \Gamma} L(\gamma, \rho) = \inf_{\gamma \in \Gamma} \int_\gamma \rho \, ds \geq 1.
$$

Proof. Let $\gamma \in \Gamma$. Clearly it is enough to restrict oneself to simple arcs $\gamma$ (i.e., with no self-intersections). Parametrize $\gamma$ as $\gamma : [0, 1] \rightarrow Q$ so that it starts on $\beta_+$. Let

$$
l = \inf\{t : 0 < t < 1, \gamma(t) \in \mathbb{R}\}.
$$

Denote by $\alpha$ the curve $\gamma|[0, l]$ with the orientation reversed, where by $\gamma|[0, l]$ we mean $\gamma$ restricted to $[0, l]$. Thus $\alpha : [0, l] \rightarrow Q \cap \{\text{Im } w \geq 0\}$ is a curve which starts on the real axis and ends on $\beta_+$. It suffices to show that $\int_\alpha \rho \, ds \geq 1/2$. Note that if the euclidean length of $\alpha$ is $\geq 1$, we are done, for $\rho \geq 1/2$ in $Q$. So we assume that the euclidean length of $\alpha$ is $< 1$. Label the components of the open set $R_1 - \mathbb{R}^*$ as $A_1, A_2, \ldots$, and those of $R_3 - \mathbb{R}^*$ as $B_1, B_2, \ldots$. Let $D = (\cup_j A_j) \cup (\cup_j B_j)$, so that $D$ is open.

If $\alpha((0, l)) \cap D = \emptyset$, then $\alpha(l) = \lim_{t \rightarrow l} t \in \beta_+ \cap \mathbb{R}^*$, and $\alpha((0, l)) \cap \{\text{Im } w = 1/2\}$.

Further $\alpha((0, l)) \subset \mathbb{R}^*$, so by (7) there is nothing to prove. So for the rest of the proof we assume $\alpha((0, l)) \cap D \neq \emptyset$. We define numbers $t_k, s_k$ and set(s) $C_k, k = 1, 2, \ldots$, as follows:

Set $s_0 = t_0 = 0, C_0 = \emptyset$, and define

$$
t_1 = \inf\{t : t > 0, \alpha((0, t)) \cap D \neq \emptyset\}.
$$

We assert that there exists a unique component $C_1$ of $D$ such that $\alpha(t_1) \in \partial C_1 \cap \beta_+^*$, here $C_1 = A_{i_0}$ for some $i_0$, or $C_1 = B_{j_0}$ for some $j_0$. Indeed it is not difficult to check that $\alpha(t_1) \in \partial D \cap \partial \mathbb{R}^* \subset \beta_+^*$. Further if $\alpha(t_1)$ lies on the boundary of two components of $D$, then $\alpha(t_1) \in \beta_+ \cap \beta_-^*$, and so $t_1 = l$, which is a contradiction. Hence $\alpha(t_1)$ determines $C_1$ uniquely.

We now define recursively, for $k \geq 1$, the following:

If $\alpha((t_k, l)) \cap (D - C_k) = \emptyset$, we set

$$
s_k = \sup\{t : t_k < t < l, \alpha(t) \in \partial C_k \cap \beta_+^*\}
$$

(9) and stop. Otherwise we define

$$
t_{k+1} = \inf\{t : t > t_k, \alpha((t, t)) \cap (D - C_k) \neq \emptyset\},
$$

$$
s_k = \sup\{t : t_k < t < t_{k+1}, \alpha(t) \in \partial C_k \cap \beta_+^*\},
$$

(10) and stop.
and \( C_{k+1} \) as the unique component of \( D \) such that \( \alpha(t_{k+1}) \in \partial C_{k+1} \). Increment \( k \) by 1 and proceed with the recursion.

The proof that \( C_{k+1} \) exists uniquely is exactly similar to the corresponding proof for \( C_1 \).

Note that \( \alpha(t_k), \alpha(s_k) \in \partial C_k \cap \beta^*_+, \) and if \( \{s_k\} \) is an infinite sequence, then \( 0 < t_1 < s_1 < t_2 < s_2 < \cdots \). If \( \{s_k\} \) is a finite sequence, say \( \{s_1, \ldots, s_n\} \), then \( 0 < t_1 < s_1 < \ldots < t_n \leq s_n \).

Let \( k \) be such that \( \alpha(t_k), \alpha(s_k) \) are defined.

We compute \( \int_{\alpha([0,s_k])} \rho \, ds \). Decompose \( \alpha([0,s_k]) \) as follows:

\[
\alpha_1 = \alpha([0,t_1]), \quad \alpha'_{1} = \alpha([t_1,s_1]), \\
\vdots \\
\alpha_k = \alpha([s_{k-1},t_k]), \quad \alpha'_{k} = \alpha([t_k,s_k]),
\]

so that

\[
\int_{\alpha([0,s_k])} \rho \, ds = \left( \int_{\alpha_1} + \int_{\alpha'_{1}} + \cdots + \int_{\alpha_k} + \int_{\alpha'_{k}} \right) \rho \, ds.
\]

Let \( \alpha(t_j) = u_j + i v_j \) and \( \alpha(s_j) = \xi_j + i \eta_j \), for \( j = 1, \ldots, k \). By the definition of \( t_j \)'s and \( s_j \)'s one sees that each \( \alpha_j \subset R^* \), so that by (7),

\[
\int_{\alpha_j} \rho \, ds \geq |\alpha(t_j) - \alpha(s_{j-1})| \geq |v_j - \eta_{j-1}|.
\]

For the \( \alpha'_j \)'s we note that if:

(i) \( C_j = B_{i_0} \), for some \( i_0 \), then \( \alpha'_j \cap (D - B_{i_0}) = \emptyset \). Thus on \( \alpha'_j, \rho \geq 1 \), and we have

\[
\int_{\alpha'_j} \rho \, ds \geq |\eta_j - v_j|.
\]

(ii) \( C_j = A_{i_0} \), for some \( i_0 \). Then \( \alpha(t_j), \alpha(s_j) \in \partial A_{i_0} \cap \beta^*_+ \), which is a Lipshitz-(1/2) curve. This means \( |\eta_j - v_j| \leq (1/2)|\xi_j - u_j| \), and since \( \rho \geq 1/2 \), this yields that

\[
\int_{\alpha'_j} \rho \, ds \geq \frac{1}{2}((\eta_j - v_j)^2 + (\xi_j - u_j)^2)^{1/2} \geq \frac{\sqrt{5}}{2} |\eta_j - v_j| \geq |\eta_j - v_j|.
\]

Now using (10), (11), (12), and (13) we get, since \( \eta_0 = 0 \), that

\[
\int_{\alpha([0,s_k])} \rho \, ds \geq \sum_{j=1}^k |v_j - \eta_{j-1}| + \sum_{j=1}^k |\eta_j - v_j| \\
\geq \sum_{j=1}^k (v_j - \eta_{j-1}) + \sum_{j=1}^k (\eta_j - v_j) \\
= \eta_k.
\]

We now consider two cases:

Case I: The sequence \( \{s_k\} \) is infinite.
We claim that in this case $\text{Im}(\alpha(s_k)) = \eta_k \to 1/2$. For if not, then we can show that $\alpha$ is nonrectifiable. Indeed, suppose by contradiction, that there exists $\tau > 0$ such that
\begin{equation}
\eta_k < 1/2 - \tau,
\end{equation}
for infinitely many values of $k$. For any such $k$ consider $\alpha(s_k) = \xi_k + i\eta_k \in \partial C_k \cap \beta_+^*$ and $\alpha(s_{k+1}) = \xi_{k+1} + i\eta_{k+1} \in \partial C_{k+1} \cap \beta_+^*$, where $\xi_{k+1} \neq \xi_k$. Without loss of generality let $\xi_{k+1} < \xi_k$. Then there exists $\xi + i/2 \in \beta_+^*$, such that $\xi_{k+1} < \xi < \xi_k$. Since $\beta_+^*$ is a Lipschitz-2 curve in general, and $\xi + i/2, \xi_k + i\eta_k \in \beta_+^*$, we have $1/2 - \eta_k < 2(\xi_k - \xi)$. Combining this with (15),
\begin{equation}
|\alpha(s_{k+1}) - \alpha(s_k)| > |\xi_{k+1} - \xi| > |\xi_k - \xi| > \tau/2,
\end{equation}
for infinitely many values of $k$. This implies $\alpha$ is nonrectifiable, which is a contradiction.

Hence $\eta_k \to 1/2$. Now by (14), for every $k$, $\int_{\alpha} \rho \, ds \geq \int_{\alpha\{[0, s_k) \}} \rho \, ds \geq \eta_k$, and since the right side tends to $1/2$, we are done in this case.

**Case II:** The sequence $\{s_k\}$ is finite, say $\{s_1, \ldots, s_n\}$.

We know by (14) that
\begin{equation}
\int_{\alpha\{[0, s_n) \}} \rho \, ds \geq \eta_n.
\end{equation}
Suppose first that $s_n = l$; then by the definition of $s_n$, (9), it is easy to see that $\eta_n = \text{Im}(\alpha(l)) = 1/2$ and by (16) we are done.

We consider next the situation $s_n < l$. All we need to do in view of (16) is to show $\int_{\alpha\{[s_n, l) \}} \rho \, ds \geq \frac{1}{2} - \eta_n$.

Now if $\alpha\{[s_n, l) \} \subset R^*$, there is nothing to prove. If $\alpha\{[s_n, l) \} \subset C_n$, then let $\xi_n + i(\eta_n + \eta)$ be the point on $\beta_+$ directly above $\xi_n + i\eta_n$. Since $\beta_+$ is a Lipschitz-1 curve, the cone $C$ with vertex at $\xi_n + i(\eta_n + \eta)$ in the $w = u + iv$-plane given by $C : \{v = -|u - \xi_n| + (\eta_n + \eta)\}$ always lies below $\beta_+$. Hence $\alpha\{[s_n, l) \}$ must intersect this cone before terminating at $\beta_+$. The shortest euclidean distance from $\xi_n + i\eta_n$ to the cone is $\eta/\sqrt{2}$. Thus if:

(a) $\alpha\{[s_n, l) \} \subset R_1$, then $\eta = 3|h_+(\xi_n)|/2$, so that
\begin{equation}
\int_{\alpha\{[s_n, l) \}} \rho \, ds \geq \rho \eta/\sqrt{2} \geq 3|h_+(\xi_n)|/4\sqrt{2} \geq |h_+(\xi_n)|/2 = \frac{1}{2} - \eta;
\end{equation}
(b) $\alpha\{[s_n, l) \} \subset R_3$, then $\eta = |h_+(\xi_n)|$ and
\begin{equation}
\int_{\alpha\{[s_n, l) \}} \rho \, ds \geq \rho \eta/\sqrt{2} = 2|h_+(\xi_n)| = \frac{1}{2} - \eta.
\end{equation}
This proves the lemma.

Using [1, Definition 4-1] and (8) we have that
\[ \frac{1}{\lambda_{R^2}(u_1, u_2)} = \lambda_Q(\Gamma) = \sup_{\rho} \frac{L^2(\Gamma, \rho)}{A(Q, \rho)} \]
where $A(Q, \rho) = \int_Q \rho^2 \, du \, dv$ and $\rho$ is nonnegative, Borel measurable and subject to the condition $0 < A(Q, \rho) < \infty$. 

Clearly $\rho$ as defined in (7) satisfies these conditions and we have from Lemma 2 that

$$\lambda_{R^*}(u_1, u_2) \leq A(Q, \rho).$$

We now compute $A(Q, \rho)$ explicitly, using (7):

$$A(Q, \rho) = A(R^* \cap Q, \rho) + A((R_1 \cup R_2) - R^*, \rho) + A((R_3 \cup R_4) - R^*, \rho)$$

$$= \left\{ u_2 - u_1 - \frac{1}{2} \int_{u_1}^{u_2} \max(h_+(u), 0) \, du + 2 \int_{u_1}^{u_2} \min(h_+(u), 0) \, du \right\}$$

$$+ \left\{ \frac{1}{2} \int_{u_1}^{u_2} \min(h_-(u), 0) \, du - 2 \int_{u_1}^{u_2} \max(h_-(u), 0) \, du \right\}$$

$$+ \left\{ \frac{1}{4} \int_{u_1}^{u_2} \frac{3}{2} \max(h_+(u), 0) \, du \right\}$$

$$- \left\{ \frac{1}{4} \int_{u_1}^{u_2} \frac{3}{2} \min(h_-(u), 0) \, du \right\}$$

$$+ \left\{ -8 \int_{u_1}^{u_2} \min(h_+(u), 0) \, du + 8 \int_{u_1}^{u_2} \max(h_-(u), 0) \, du \right\}$$

$$= u_2 - u_1 - \frac{1}{8} \int_{u_1}^{u_2} \max(h_+(u), 0) \, du - 6 \int_{u_1}^{u_2} \min(h_+(u), 0) \, du$$

$$+ \frac{1}{8} \int_{u_1}^{u_2} \min(h_-(u), 0) \, du + 6 \int_{u_1}^{u_2} \max(h_-(u), 0) \, du.$$  

From (6) and (17) we see that $A(Q, \rho) \geq u_2 - u_1 + o(1)$. By the hypothesis (3), both $\int_{u_1}^{u_2} \max(h_-(u), 0) \, du$ and $\int_{u_1}^{u_2} - \min(h_+(u), 0) \, du \to 0$ as $u_2 > u_1 \to +\infty$. Hence

$$0 \geq -\frac{1}{8} \int_{u_1}^{u_2} \max(h_+(u), 0) \, du + \frac{1}{8} \int_{u_1}^{u_2} \min(h_-(u), 0) \, du \geq o(1),$$

which readily implies (4) and hence proves the theorem.

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