PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 4, April 1995

TWO RESULTS ON THE 2-LOCAL *EHP* SPECTRAL SEQUENCE

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(Communicated by Thomas Goodwillie)

ABSTRACT. The E_2-term of the 2-local EHP spectral sequence is shown to be a $\mathbb{Z}/2$ module. 4 is the order of the identity map on the double loop space of the fiber $W(n)$ of the double suspension $E^2: S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$.

1. INTRODUCTION

Restrict attention to the category of 2-local spaces. The EHP fibrations [6, 1]

$$\Omega^2 S^{2q+1} \xrightarrow{P} S^q \xrightarrow{E} \Omega S^{q+1} \xrightarrow{H} \Omega S^{2q+1}$$

give a tower of fibrations converging to $Q(S^0)$, whose homotopy spectral sequence is the *EHP spectral sequence* [9] $E_1^{p,q} = \pi_{p+q}(S^{2q-1}) \Rightarrow \pi_p^s$, with differentials $d_r: E_r^{p,q} \longrightarrow E_r^{p-r,q-1}$. James [6] proved that $2^{2n}\pi_*(S^{2n+1}) = 0$, by showing that $2\pi_*(S^{2n+1}) \subset \mathrm{Im}(E)$ and $E(2\pi_*(S^{2n})) \subset \mathrm{Im}(E^2)$. Thus the E_∞-term of the EHP spectral sequence is a $\mathbb{Z}/2$ module. James's work was translated to spaces [1, §5]:

Lemma 1.1. (1) $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1} \xrightarrow{2} \Omega^2 S^{4n+1}$ *is nullhomotopic.*

(2) *There exists a map* ϕ *making the following diagram homotopy commutative.*

$$\begin{array}{ccccc}
\Omega^2 S^{2n} & \xrightarrow{2} & \Omega^2 S^{2n} & \xrightarrow{\Omega^2 E} & \Omega^3 S^{2n+1} \\
 & \searrow^{\phi} & & & \uparrow{\scriptstyle \Omega E^2} \\
 & & & & \Omega S^{2n-1}
\end{array}$$

Mahowald [8] made the following conjecture, which will follow from Lemma 1.1 and an extensive amount of diagram chasing.

Received by the editors June 8, 1993.
1991 *Mathematics Subject Classification*. Primary 55Q40, 55Q15, 55Q25.
Key words and phrases. Exponents, EHP fibrations and spectral sequence, $W(n)$, coliftings.
Research supported by the NSF.

Theorem 1.2. *The E_2-term of the EHP spectral sequence is a $\mathbb{Z}/2$ module. The 4^{th} power map of $\Omega^2 W(n)$ is nullhomotopic, and $\pi_*(W(n))$ has exponent* 4.

Let $d_1^+ : \Omega^3 S^{4n+1} \xrightarrow{\Omega P} \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$ and $d_1^- : \Omega^3 S^{4n+3} \xrightarrow{\Omega P} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{4n+1}$ denote the composites which realize the first EHP differential. Selick [11] improved the James exponent to $2^{2n-[n/2]}$. Cohen [1, §6] reformulated this as a compression of the H-space squaring map on $\Omega^4 S^{4n+1}$ through $\Omega^2 S^{4n-1}$. Theorem 1.2 is implied by the following compression result, which extends their work.

Theorem 1.3. *There exist maps $\mathscr{F}^+ : \Omega^2 S^{4n-1} \longrightarrow \Omega^4 S^{4n+1}$ and $\mathscr{F}^- : \Omega^2 S^{4n-3} \longrightarrow \Omega^2 W(n)$ making the following diagrams homotopy commutative.*

$$\begin{array}{ccc} \Omega^4 S^{4n+1} & \xrightarrow{\Omega d_1^+} & \Omega^2 S^{4n-1} \\ & {\scriptstyle 2}\searrow & \downarrow {\scriptstyle \mathscr{F}^+} \\ & & \Omega^4 S^{4n+1} \end{array} \qquad \begin{array}{ccc} \Omega^4 S^{4n-1} & \xrightarrow{\Omega d_1^-} & \Omega^2 S^{4n-3} \\ {\scriptstyle 2}\downarrow & & \downarrow {\scriptstyle \mathscr{F}^-} \\ \Omega^4 S^{4n-1} & \xrightarrow{\Omega^2 \partial} & \Omega^2 W(n) \end{array}$$

2. Proofs

For any space X, we will denote by $2 : \Omega X \longrightarrow \Omega X$ the H-space squaring map. We will often use the following fact. For any map $f : \Omega X \longrightarrow \Omega Y$, the composites $\Omega^2 X \xrightarrow{\Omega f} \Omega^2 Y \xrightarrow{2} \Omega^2 Y$ and $\Omega^2 X \xrightarrow{2} \Omega^2 X \xrightarrow{\Omega f} \Omega^2 Y$ are homotopic. We will use the following result about *colifings*, which we state without proof.

Lemma 2.1. *Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fibration, and let $f : B \longrightarrow X$ be a map such that $f \cdot p : E \longrightarrow X$ is nullhomotopic. Then Ωf factors through ∂, by a* colifting $\mathscr{B} : F \longrightarrow \Omega X$, *which makes the following diagram commutes up to homotopy.*

$$\begin{array}{ccc} \Omega B & \xrightarrow{\partial} & F \\ & {\scriptstyle \Omega f}\searrow \quad \swarrow{\scriptstyle \mathscr{B}} & \\ & \Omega X & \end{array}$$

Proof of Theorem 1.3. The EHP fibration $\Omega S^{2n} \xrightarrow{\Omega E} \Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1}$ and Lemmas 1.1(1) and 2.1 give a colifting $\mathscr{B} : \Omega S^{2n} \longrightarrow \Omega^3 S^{4n+1}$ making the diagram

$$\begin{array}{ccc} \Omega^3 S^{4n+1} & \xrightarrow{\Omega P} & \Omega S^{2n} \\ & {\scriptstyle 2}\searrow \quad \swarrow{\scriptstyle \mathscr{B}} & \\ & \Omega^3 S^{4n+1} & \end{array}$$

homotopy commutative. But $\mathscr{B} \cdot E : S^{2n-1} \longrightarrow \Omega^3 S^{4n+1}$ is nullhomotopic. By Lemma 2.1 and the EHP fibration $\Omega^2 S^{4n-1} \xrightarrow{P} S^{2n-1} \xrightarrow{E} \Omega S^{2n}$, there exists a colifting $\mathscr{F}^+\colon \Omega^2S^{4n-1} \longrightarrow \Omega^4S^{4n+1}$ making the diagram

$$\begin{array}{ccc}
\Omega^2S^{2n} & \xrightarrow{\quad\Omega H\quad} & \Omega^2S^{4n-1} \\
 & {\scriptstyle\Omega\mathscr{B}}\searrow \qquad \swarrow{\scriptstyle\mathscr{F}^+} & \\
 & \Omega^4S^{4n+1} &
\end{array}$$

homotopy commutative. This proves the first part of Theorem 1.3.

By Lemma 1.1(2), the composite $\Omega^2S^{2n} \xrightarrow{2-\Omega E\cdot\phi} \Omega^2S^{2n} \xrightarrow{\Omega^2E} \Omega^3S^{2n+1}$ is nullhomotopic. Hence there exists a map $\psi\colon \Omega^2S^{2n} \longrightarrow \Omega^4S^{4n+1}$ making the diagram

$$\begin{array}{ccccc}
 & \Omega^2S^{2n} & \xrightarrow{\qquad\Omega H\qquad} & \Omega^2S^{4n-1} & \\
{\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle 2-\Omega E\cdot\phi} & & \searrow{\scriptstyle 2} \\
\Omega^4S^{4n+1} & \xrightarrow{\qquad\Omega^2P\qquad} & \Omega^2S^{2n} & \xrightarrow{\qquad\Omega H\qquad} & \Omega^2S^{4n-1}
\end{array}$$

commute up to homotopy, since (cf. [1, Proof of Lemma 4.1]) ΩH is linear. We have an induced map of fibers $\beta\colon \Omega S^{2n-1} \longrightarrow \Omega W(n)$, obtained by pulling back the outer trapezoid to the left, making the following diagram homotopy commutative.

$$\begin{array}{ccccccc}
\Omega^3S^{4n-1} & \xrightarrow{\Omega P} & \Omega S^{2n-1} & \xrightarrow{\Omega E} & \Omega^2S^{2n} & \xrightarrow{\Omega H} & \Omega^2S^{4n-1} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle 2} \\
\Omega^3S^{4n-1} & \xrightarrow{\Omega\partial} & \Omega W(n) & \xrightarrow{\Omega j} & \Omega^4S^{4n+1} & \xrightarrow{\Omega d_1^+} & \Omega^2S^{4n-1}
\end{array}$$

But $\beta\cdot E\colon S^{2n-2} \longrightarrow \Omega W(n)$ is nullhomotopic. The *EHP* fibration $\Omega^2S^{4n-3} \xrightarrow{P} S^{2n-2} \xrightarrow{E} \Omega S^{2n-1}$ and Lemma 2.1 then yield the colifting $\mathscr{F}^-\colon \Omega^2S^{4n-3} \longrightarrow \Omega^2W(n)$ making the following diagram homotopy commutative.

$$\begin{array}{ccc}
\Omega^2S^{2n-1} & \xrightarrow{\quad\Omega H\quad} & \Omega^2S^{4n-3} \\
 & {\scriptstyle\Omega\beta}\searrow \qquad \swarrow{\scriptstyle\mathscr{F}^-} & \\
 & \Omega^2W(n) &
\end{array}$$

□

Proof of Theorem 1.2. By Theorem 1.3, $\mathrm{Ker}(d_1^+)_* \subset \pi_*(S^{4n+1})$ is a $\mathbb{Z}/2$ module. Thus each $E_2^{*,2n+1}$ is a $\mathbb{Z}/2$ module. By Theorem 1.3, any cycle $\alpha \in \mathrm{Ker}(d_1^-)_* \subset \pi_*(S^{4n-1})$ satisfies $2\alpha \in \mathrm{Im}(d_1^+)_*$. Hence each $E_2^{*,2n}$ is a $\mathbb{Z}/2$ module.

We have the fibration sequence $\Omega^2S^{4n-1} \xrightarrow{\partial} W(n) \xrightarrow{j} \Omega^3S^{4n+1} \xrightarrow{d_1^+} \Omega S^{4n-1}$. By Theorem 1.3, the composite $\Omega W(n) \xrightarrow{\Omega j} \Omega^4S^{4n+1} \xrightarrow{2} \Omega^4S^{4n+1}$ is nullhomotopic. As indicated by the following homotopy commutative diagram, there exists a lifting $f\colon \Omega W(n) \longrightarrow \Omega^3S^{4n-1}$ of the H space squaring map of $\Omega W(n)$

through $\Omega\partial$.

$$\begin{array}{ccccccc}
\Omega^3 S^{4n-1} & \xrightarrow{\Omega\partial} & \Omega W(n) & \xrightarrow{\Omega j} & \Omega^4 S^{4n+1} & & \\
& \nwarrow{\scriptstyle f} & \uparrow{\scriptstyle 2} & \nearrow{\scriptstyle *} & \uparrow{\scriptstyle 2} & \nwarrow{\scriptstyle \mathscr{S}^{+}} & \\
& & \Omega W(n) & \xrightarrow{\Omega j} & \Omega^4 S^{4n+1} & \xrightarrow{\Omega d_1^{+}} & \Omega^2 S^{4n-1}
\end{array}$$

We have the following homotopy commutative diagrams.

$$\begin{array}{ccc}
\Omega^2 S^{4n-1} & \xrightarrow{P} & S^{2n-1} \\
& {\scriptstyle \partial}\searrow \quad \nearrow{\scriptstyle \iota} & \\
& W(n) &
\end{array}$$

$$\begin{array}{ccccc}
& & \Omega^3 S^{4n-1} & \xrightarrow{d_1^{-}} & \Omega S^{4n-3} \\
& {\scriptstyle f}\nearrow & \downarrow{\scriptstyle \Omega\partial} & \searrow{\scriptstyle \Omega P} & \nearrow{\scriptstyle H} \\
\Omega W(n) & \xrightarrow{2} & \Omega W(n) & \xrightarrow{\Omega\iota} & \Omega S^{2n-1}
\end{array}$$

By looping the above parallelogram and applying Lemma 1.1(1), we see that the composite $\Omega^2 W(n) \xrightarrow{\Omega f} \Omega^4 S^{4n-1} \xrightarrow{\Omega d_1^{-}} \Omega^2 S^{4n-3}$ is nullhomotopic. The composite $\Omega^2 W(n) \xrightarrow{\Omega f} \Omega^4 S^{4n-1} \xrightarrow{2} \Omega^4 S^{4n-1} \xrightarrow{\Omega^2\partial} \Omega^2 W(n)$ is nullhomotopic, by Theorem 1.3. Hence $4\colon \Omega^2 W(n) \longrightarrow \Omega^2 W(n)$, the 4^{th} power map, is nullhomotopic. □

3. Remarks

James [6] also showed that $2E(x) = 0$ for all $x \in \operatorname{Ker}(E^2) \subset \pi_*(S^q)$. When $q = 2n-1$, this gives evidence for Theorem 1.2, as it is implied by $4\pi_*(W(n)) = 0$. We used the case $q = 2n$ of James's result in an earlier version of our paper.

Richter [10] strengthened Theorem 1.3, showing that $2 \simeq -\Omega E^2 \cdot d_1^{+}$ on $\Omega^3 S^{4n+1}$ and $2 - \Omega^3(2\iota) \simeq -\Omega E^2 \cdot d_1^{-}$ on $\Omega^3 S^{4n-1}$, solving a conjecture of Gray [3] and Mahowald, which Harper [5] proved at odd primes. At an odd prime p, Cohen, Moore, and Neisendorfer [2] showed that the p^{th} power map on $\Omega W(n)$ is nullhomotopic. Gray [4] showed that $W(n)$ deloops, essentially by delooping the map d_1^{+}. It was already known that $\pi_*(W(2))$ had exponent 4, by Cohen's [1, Theorem 19.1] splitting $\Omega^2 S^5\{2\} \simeq W(2) \times \Omega^2 S^3\langle 3\rangle$.

Mahowald [8] further conjectured that $(d_1^{-})_* = 0\colon \pi_{*+2}(S^{4n-1}) \longrightarrow \pi_*(S^{4n-3})$. Note that James shows that $\operatorname{Ker}(E) \subset \pi_*(S^{2n+1})$ is a $\mathbb{Z}/4$ module.

The conjecture implies that $\mathrm{Ker}(E)$ is a $\mathbb{Z}/2$ module. By [10], the conjecture also implies

Conjecture C2. *For any element* $\alpha \in \pi_*(S^{4n-1})$, $(2\iota)\cdot\alpha = 2\alpha \in \pi_*(S^{4n-1})$.

One might wonder whether $2 \simeq \Omega^k(2\iota)$ on $\Omega^k S^{4n-1}$ for some k. Note [1, §§11 and 12] that away from Arf invariant one or Hopf invariant one dimensions, k must be at least 3.

Acknowledgment

Thanks to Paul Burchard for the commutative diagram package, which uses $\mathcal{LA}_{\mathcal{M}}\mathcal{S}$-TEX fonts.

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