TWO RESULTS ON THE 2-LOCAL EHP SPECTRAL SEQUENCE

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ABSTRACT. The E_2 -term of the 2-local EHP spectral sequence is shown to be a $\mathbb{Z}/2$ module. 4 is the order of the identity map on the double loop space of the fiber W(n) of the double suspension $E^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$.

1. Introduction

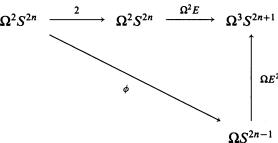
Restrict attention to the category of 2-local spaces. The EHP fibrations [6, 1]

$$\Omega^2 S^{2q+1} \xrightarrow{P} S^q \xrightarrow{E} \Omega S^{q+1} \xrightarrow{H} \Omega S^{2q+1}$$

give a tower of fibrations converging to $Q(S^0)$, whose homotopy spectral sequence is the *EHP* spectral sequence [9] $E_1^{p,q} = \pi_{p+q}(S^{2q-1}) \Rightarrow \pi_p^s$, with differentials $d_r: E_r^{p,q} \to E_r^{p-r,q-1}$. James [6] proved that $2^{2n}\pi_*(S^{2n+1}) = 0$, by showing that $2\pi_*(S^{2n+1}) \subset \text{Im}(E)$ and $E(2\pi_*(S^{2n})) \subset \text{Im}(E^2)$. Thus the E_{∞} -term of the *EHP* spectral sequence is a $\mathbb{Z}/2$ module. James's work was translated to spaces [1, §5]:

Lemma 1.1. (1) $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1} \xrightarrow{2} \Omega^2 S^{4n+1}$ is nullhomotopic.

(2) There exists a map ϕ making the following diagram homotopy commutative.



Mahowald [8] made the following conjecture, which will follow from Lemma 1.1 and an extensive amount of diagram chasing.

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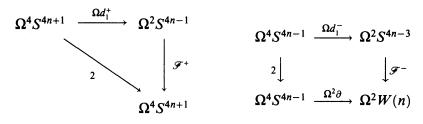
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Theorem 1.2. The E_2 -term of the EHP spectral sequence is a $\mathbb{Z}/2$ module. The 4^{th} power map of $\Omega^2W(n)$ is nullhomotopic, and $\pi_*(W(n))$ has exponent 4.

Let $d_1^+\colon \Omega^3 S^{4n+1} \xrightarrow{\Omega P} \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$ and $d_1^-\colon \Omega^3 S^{4n+3} \xrightarrow{\Omega P} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{4n+1}$ denote the composites which realize the first EHP differential. Selick [11] improved the James exponent to $2^{2n-[n/2]}$. Cohen [1, §6] reformulated this as a compression of the H-space squaring map on $\Omega^4 S^{4n+1}$ through $\Omega^2 S^{4n-1}$. Theorem 1.2 is implied by the following compression result, which extends their work.

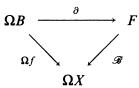
Theorem 1.3. There exist maps $\mathscr{F}^+: \Omega^2 S^{4n-1} \to \Omega^4 S^{4n+1}$ and $\mathscr{F}^-: \Omega^2 S^{4n-3} \to \Omega^2 W(n)$ making the following diagrams homotopy commutative.



2. Proofs

For any space X, we will denote by $2: \Omega X \to \Omega X$ the H-space squaring map. We will often use the following fact. For any map $f: \Omega X \to \Omega Y$, the composites $\Omega^2 X \xrightarrow{\Omega f} \Omega^2 Y \xrightarrow{2} \Omega^2 Y$ and $\Omega^2 X \xrightarrow{2} \Omega^2 X \xrightarrow{\Omega f} \Omega^2 Y$ are homotopic. We will use the following result about *coliftings*, which we state without proof.

Lemma 2.1. Let $F \xrightarrow{1} E \xrightarrow{p} B$ be a fibration, and let $f: B \to X$ be a map such that $f \cdot p: E \to X$ is nullhomotopic. Then Ωf factors through ∂ , by a colifting $\mathscr{B}: F \to \Omega X$, which makes the following diagram commutes up to homotopy.



Proof of Theorem 1.3. The EHP fibration $\Omega S^{2n} \xrightarrow{\Omega E} \Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1}$ and Lemmas 1.1(1) and 2.1 give a colifting $\mathscr{B}: \Omega S^{2n} \longrightarrow \Omega^3 S^{4n+1}$ making the diagram

$$\Omega^{3}S^{4n+1} \xrightarrow{\Omega P} \Omega S^{2n}$$

$$\Omega^{3}S^{4n+1} \xrightarrow{\mathbb{Z}^{2}} \Omega^{3}S^{4n+1}$$

homotopy commutative. But $\mathscr{B} \cdot E \colon S^{2n-1} \to \Omega^3 S^{4n+1}$ is nullhomotopic. By Lemma 2.1 and the EHP fibration $\Omega^2 S^{4n-1} \xrightarrow{P} S^{2n-1} \xrightarrow{E} \Omega S^{2n}$, there exists

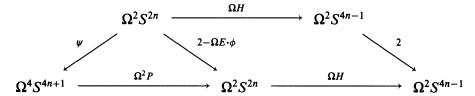
a colifting $\mathcal{F}^+: \Omega^2 S^{4n-1} \to \Omega^4 S^{4n+1}$ making the diagram

$$\Omega^2 S^{2n} \xrightarrow{\Omega H} \Omega^2 S^{4n-1}$$

$$\Omega^4 S^{4n+1} \xrightarrow{\mathcal{F}^+}$$

homotopy commutative. This proves the first part of Theorem 1.3.

By Lemma 1.1(2), the composite $\Omega^2 S^{2n} \xrightarrow{2-\Omega E \cdot \phi} \Omega^2 S^{2n} \xrightarrow{\Omega^2 E} \Omega^3 S^{2n+1}$ is nullhomotopic. Hence there exists a map $\psi \colon \Omega^2 S^{2n} \longrightarrow \Omega^4 S^{4n+1}$ making the diagram



commute up to homotopy, since (cf. [1, Proof of Lemma 4.1]) ΩH is linear. We have an induced map of fibers $\beta \colon \Omega S^{2n-1} \to \Omega W(n)$, obtained by pulling back the outer trapezoid to the left, making the following diagram homotopy commutative.

$$\Omega^{3}S^{4n-1} \xrightarrow{\Omega P} \Omega S^{2n-1} \xrightarrow{\Omega E} \Omega^{2}S^{2n} \xrightarrow{\Omega H} \Omega^{2}S^{4n-1}$$

$$\downarrow^{2} \qquad \qquad \downarrow^{\beta} \qquad \qquad \downarrow^{\psi} \qquad \qquad \downarrow^{2}$$

$$\Omega^{3}S^{4n-1} \xrightarrow{\Omega \partial} \Omega W(n) \xrightarrow{\Omega j} \Omega^{4}S^{4n+1} \xrightarrow{\Omega d_{1}^{+}} \Omega^{2}S^{4n-1}$$

But $\beta \cdot E \colon S^{2n-2} \longrightarrow \Omega W(n)$ is nullhomotopic. The *EHP* fibration $\Omega^2 S^{4n-3} \xrightarrow{P} S^{2n-2} \xrightarrow{E} \Omega S^{2n-1}$ and Lemma 2.1 then yield the colifting \mathscr{F}^- : $\Omega^2 S^{4n-3} \longrightarrow \Omega^2 W(n)$ making the following diagram homotopy commutative.

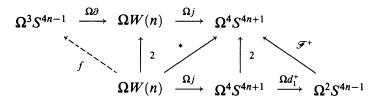
$$\Omega^2 S^{2n-1} \xrightarrow{\Omega H} \Omega^2 S^{4n-3}$$

$$\Omega^2 W(n)$$

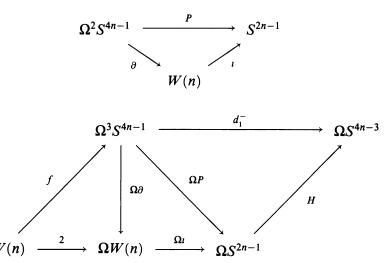
Proof of Theorem 1.2. By Theorem 1.3, $\operatorname{Ker}(d_1^+)_* \subset \pi_*(S^{4n+1})$ is a $\mathbb{Z}/2$ module. Thus each $E_2^{*,\,2n+1}$ is a $\mathbb{Z}/2$ module. By Theorem 1.3, any cycle $\alpha \in \operatorname{Ker}(d_1^-)_* \subset \pi_*(S^{4n-1})$ satisfies $2\alpha \in \operatorname{Im}(d_1^+)_*$. Hence each $E_2^{*,\,2n}$ is a $\mathbb{Z}/2$ module.

We have the fibration sequence $\Omega^2 S^{4n-1} \xrightarrow{\partial} W(n) \xrightarrow{j} \Omega^3 S^{4n+1} \xrightarrow{d_1^+} \Omega S^{4n-1}$. By Theorem 1.3, the composite $\Omega W(n) \xrightarrow{\Omega j} \Omega^4 S^{4n+1} \xrightarrow{2} \Omega^4 S^{4n+1}$ is nullhomotopic. As indicated by the following homotopy commutative diagram, there exists a lifting $f: \Omega W(n) \to \Omega^3 S^{4n-1}$ of the H space squaring map of $\Omega W(n)$

through $\Omega \partial$.



We have the following homotopy commutative diagrams.



By looping the above parallelogram and applying Lemma 1.1(1), we see that the composite $\Omega^2W(n) \xrightarrow{\Omega f} \Omega^4S^{4n-1} \xrightarrow{\Omega d_1^-} \Omega^2S^{4n-3}$ is nullhomotopic. The composite $\Omega^2W(n) \xrightarrow{\Omega f} \Omega^4S^{4n-1} \xrightarrow{2} \Omega^4S^{4n-1} \xrightarrow{\Omega^2\partial} \Omega^2W(n)$ is nullhomotopic, by Theorem 1.3. Hence 4: $\Omega^2W(n) \to \Omega^2W(n)$, the 4th power map, is nullhomotopic. \square

3. Remarks

James [6] also showed that 2E(x)=0 for all $x\in \mathrm{Ker}(E^2)\subset \pi_*(S^q)$. When q=2n-1, this gives evidence for Theorem 1.2, as it is implied by $4\pi_*(W(n))=0$. We used the case q=2n of James's result in an earlier version of our paper.

Richter [10] strengthened Theorem 1.3, showing that $2 \simeq -\Omega E^2 \cdot d_1^+$ on $\Omega^3 S^{4n+1}$ and $2 - \Omega^3(2i) \simeq -\Omega E^2 \cdot d_1^-$ on $\Omega^3 S^{4n-1}$, solving a conjecture of Gray [3] and Mahowald, which Harper [5] proved at odd primes. At an odd prime p, Cohen, Moore, and Neisendorfer [2] showed that the p^{th} power map on $\Omega W(n)$ is nullhomotopic. Gray [4] showed that W(n) deloops, essentially by delooping the map d_1^+ . It was already known that $\pi_*(W(2))$ had exponent 4, by Cohen's [1, Theorem 19.1] splitting $\Omega^2 S^5\{2\} \simeq W(2) \times \Omega^2 S^3\langle 3 \rangle$.

Mahowald [8] further conjectured that $(d_1^-)_* = 0$: $\pi_{*+2}(S^{4n-1}) \to \pi_*(S^{4n-3})$. Note that James shows that $Ker(E) \subset \pi_*(S^{2n+1})$ is a $\mathbb{Z}/4$ module.

The conjecture implies that Ker(E) is a $\mathbb{Z}/2$ module. By [10], the conjecture also implies

Conjecture C2. For any element $\alpha \in \pi_*(S^{4n-1})$, $(2i) \cdot \alpha = 2\alpha \in \pi_*(S^{4n-1})$.

One might wonder whether $2 \simeq \Omega^k(2i)$ on $\Omega^k S^{4n-1}$ for some k. Note [1, §§11 and 12] that away from Arf invariant one or Hopf invariant one dimensions, k must be at least 3.

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