ABSTRACT. Darboux's lemma states that a closed nondegenerate two-form Ω, defined on an open set in \( \mathbb{R}^{2n} \) (or in a 2n-dimensional manifold), can locally be given the form \( \sum dq_i \wedge dp_i \), in suitable coordinates, traditionally denoted by \( q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n \). There is an elegant proof by J. Moser and A. Weinstein. The author has presented a proof that was extracted from Carathéodory's book on Calculus of Variations. Carathéodory works with a (local) "integral" of \( \Omega \), that is, with a one-form \( \alpha \) satisfying \( d\alpha = \Omega \). It turns out that the proof becomes much more transparent if one works with \( \Omega \) itself.

As in [3] we start by writing \( \Omega \) (locally) as \( \sum_{i=1}^{N} df_i \wedge dg_i \), with some functions \( f_1, f_2, \ldots, f_N, g_1, g_2, \ldots, g_N \), and with \( N \geq n \) of course. (For this step we take an integral \( \alpha \) of \( \Omega \) and write it as \( \sum_{i=1}^{N} f_i df_i \).) We now try to reduce \( N \), if it is larger than \( n \).

Since \( \Omega^n \) is not 0, some \( n \) of the terms in the sum for \( \Omega \) must have nonzero exterior product; the corresponding \( f \)'s and \( g \)'s can then be taken as coordinates \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \) and we can write \( \Omega = \sum_{i=1}^{n} du_i \wedge dv_i + \sum_{i=n+1}^{N} df_j \wedge dg_j \). To this situation we apply a standard classical and basic proposition of Hamiltonian transformation theory [5].

Let \( \omega \) be a closed nondegenerate two-form on an open set in a manifold \( M^{2n} \) (with local coordinates \( x \), when needed), and let \( H \) be a "time-dependent Hamiltonian", i.e., a function \( H(x, t) \) on \( M \times \mathbb{R} \) (or on a suitable open subset thereof). Write \( \omega_H \) for the two-form \( \omega - dH \wedge dt \) (here \( \omega \) has been pulled back to \( M \times \mathbb{R} \) and \( t \) is the standard coordinate on \( \mathbb{R} \)).

**Proposition.** There exists a (local) diffeomorphism \( F \) of \( M \times \mathbb{R} \) "over \( \mathbb{R} \)"., i.e., of the form \( x' = F(x, t), t' = t \) (or, briefly, of the form \( x' = F(x, t) \)) with inverse \( x = G(x', t) \) such that

\[
F^* \omega_H = \omega \quad \text{and} \quad G^* \omega = \omega_H.
\]

One says that "\( H \) has been reduced to 0 by \( F \)." As a matter of fact, \( F \) is simply the expression for the solutions of the associated "canonical equations" in terms of the initial values for \( t = 0 \). The proof is a simple computation; we bring it, for completeness, at the end.
We apply the proposition to $\Omega$ above by using $\sum du_i \wedge dv_i$ as $\omega$ and $-f_N$ as $H$. Thus we have functions $u'_i = \phi_i(u_j, v_j, t)$, $v'_i = \psi_i(u_j, v_j, t)$ such that the equation
\[
\sum du_i \wedge dv_i + df_N \wedge dt = \sum d\phi_i \wedge d\psi_i
\]
holds identically in $(u_j, v_j, t)$. We now substitute the function $g_N$, from the expression for $\Omega$, for $f$ in this equation (i.e., we take the pullback of the equation under the embedding of $M$ into $M \times \mathbb{R}$ via $x \mapsto (x, g_N(x))$). Thus $\sum du_i \wedge dv_i + df_N \wedge dg_N$ equals $\sum d\Phi_i \wedge d\Psi_i$, where $\Phi_i(u_j, v_j)$ means $\phi_i(u_j, v_j, g_N(u_j, v_j))$ and similarly for $\Psi_i$. So $\Omega = \sum du_i \wedge dv_i + \sum_{n+1} d\phi_i \wedge d\phi_{i+1}$ equals $\sum d\phi_i \wedge d\psi_i + \sum_{n+1} d\psi_i \wedge d\psi_{i+1}$, and so the number of terms in the expression for $\Omega$ has been reduced by 1. Darboux's lemma follows by iteration. □

Now we prove the proposition. We express the usual canonical differential equations of Hamiltonian theory in the language of exterior forms: A vector field $\tilde{X}$ on $M \times \mathbb{R}$ (or on an open subset thereof) will be called Hamiltonian (to $H$) if
(a) it is of the form $(X, \partial_t)$, where $\partial_t$ is the standard vector field $\mathbb{R}$ (thus $\partial_t f = f'$) and where $X$ at any point $(x, t)$ is tangent to $M \times \mathbb{R}$, so that $X$ is a "time-dependent vector field" on $M$; and
(b) the substitution operator $i_{\tilde{X}}$ nullifies the form $\omega_H = \omega - dH \wedge dt$.

(For any vector field $Y$ the operator $i_Y$ operates on an exterior form $\pi$ by substituting $Y$ into the first slot to $\pi$. It is characterized by three properties: (1) it nullifies functions (i.e., 0-forms); (2) one has $i_Y dh = dh(Y) = Y.h$ for any function $h$; (3) it is a (graded) derivation: $i_Y(\Lambda^p) = i_Y \Lambda^p + (-1)^{deg \Lambda^p} \Lambda^p i_Y$.)

We split the differential $dH$ into its $M$- and $\mathbb{R}$-components (defined by restriction to the $M$- or $\mathbb{R}$-factor at $(x, t)$); we write this as $dH = dMH + Hdt$. The Hamiltonian condition $i_{\tilde{X}} \omega_H = 0$, i.e., $i_{\tilde{X}} \omega = (i_{\tilde{X}} dH) dt - dH$, means then $i_{\tilde{X}} \omega = -dMH$ and $i_{\tilde{X}} dH = (\tilde{X}, H) = H_t$; the second relation can also be written as $X.H = 0$ or $dMH(X) = 0$ and is a consequence of the first, since $\omega$ is skewsymmetric, and so $-dMH(X) = i_X \omega(X) = \omega(X, X) = 0$. Since $\omega$ is nondegenerate, the relation $i_{\tilde{X}} \omega = -dMH$ shows that the Hamiltonian field $\tilde{X}$ exists and is unique. For the case $\omega = \sum dp_i \wedge dq_i$ the relation $i_{\tilde{X}} \omega = -dMH$ amounts to the canonical equations $\dot{q}_i = H_{q_i}, \dot{p}_i = -H_{p_i}$.

We now construct the map $F$ of the proposition: as noted after the proposition, it simply sends each line $x \times \mathbb{R}$ to the trajectory of $\tilde{X}$ through $(x, 0)$. (In particular, we have $x = F(x, 0)$.) This is a diffeomorphism by standard theorems about ordinary differential equations. Clearly the vector fields $\partial_t$ on $M \times \mathbb{R}$ map to $X$ under $F$. It follows that $i_{\partial_t}$ nullifies $F^* \omega_H$.

We write $F^* \omega_H$ as $\omega_0 + \beta \wedge dt$, where $\omega_0$ and $\beta$ are nullified by $\partial_t$, i.e., do not involve any $dt$. The relation $i_{\partial_t} F^* \omega_H = 0$ then says $\beta = 0$; so we have $F^* \omega_H = \omega_0$. Since $\omega_H$ is closed, so is $\omega_0$; the equation $d\omega_0 = 0$ implies that the $t$-derivatives of the coefficients $a_{ij}$ of $\omega_0 = \sum a_{ij} dx_i \wedge dx_j$ vanish and that the form $\omega_0$ does not depend on $t$ (for this the domain of definition should be convex in the $t$-direction and connected). Thus $F^* \omega_H$ is simply a two-form on $M$, pulled back to $M \times \mathbb{R}$; and finally, since the map $F$ is the identity on the slice $t = 0$, $F^* \omega_H$ equals $\omega$. □
REFERENCES


DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305

E-mail address: samelson@gauss.stanford.edu