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WEIGHTED INEQUALITIES FOR CONVOLUTIONS

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ABSTRACT. For certain convolution operators T on R^+ or R^n, sufficient conditions are given which ensure that T is bounded between weighted Lebesgue spaces. The class of operators considered includes many of classical interest; in particular, new inequalities are obtained for the Laplace transform, the Poisson integral on $R^n \times R^+$, and Goldberg's transform.

1. INTRODUCTION

In a recent paper Bloom [2] obtained weighted inequalities for the Laplace transform $\mathscr{L}$ using known weighted inequalities for the Hardy operator. The purpose of this paper is twofold. First, we show that these weighted Hardy inequalities may be used in a different and, we think, more elementary way to obtain weighted inequalities for a class of operators on $R^+ = (0, \infty)$. Our results will be seen to include and sharpen Bloom's results for $\mathscr{L}$. Secondly, we show that our approach admits a natural extension to a class of operators on R^n; applications will be illustrated by deriving new inequalities for Goldberg's transform on R and the Poisson integral on the half space $R^{n+} = R^n \times R^+$.

If X denotes R^+ or R^n and μ is a positive Borel measure on X, denote by $L^p(X, \mu)$ the weighted Lebesgue space of measurable functions f on X for which $||f||_{p,\mu} < \infty$ where

$$||f||_{p,\mu} = \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \mu \operatorname{ess.sup}_{y \in X} |f(y)| & \text{if } p = \infty. \end{cases}$$

If $f(y)$ is expressed as a formula in y it will sometimes be convenient to slightly abuse the notation $||f||_{p,\mu}$ by writing instead $||f(y)||_{p,\mu}$.

The operators considered here are convolutions with a suitable kernel k and are given by

$$Tf(x) = \begin{cases} \int_{R^+} k(x/y) f(y) \, dy/y & \text{if } X = R^+, \\ \int_{R^n} k(|x-y|) f(y) \, dy & \text{if } X = R^n. \end{cases} \tag{1.1}$$

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Given such T and exponents $1 \le p, q \le \infty$, sufficient conditions to be satisfied by positive Borel measures μ and ν are given which ensure that

$$\|Tf\|_{q,\mu} \le C\|f\|_{p,\nu} \tag{1.2}$$

for a constant C independent of $f \in L^p(X, \nu)$. Since $\nu = \nu_a + \nu_s$ where ν_a is absolutely continuous and ν_s is singular with respect to Lebesgue measure on X, there is no loss of generality in taking $\nu = \nu_a$ since the left side of (1.2) is unchanged if f is redefined to be zero on the support of ν_s. For convenience we set $v = d\nu_a/dx$ and $d\sigma/dx = v^{-1/(p-1)}$. As usual p' denotes the conjugate exponent of p given by $1/p + 1/p' = 1$ and if $q < p$ the exponent r is defined by $1/r = 1/q - 1/p$. We define a function $\omega(t) = \omega(X; \mu, \nu; q, p; t)$ for $t > 0$ as follows, χ_E denoting the characteristic function of E.

If $X = R^+$,

$$\omega(t) = \begin{cases} \sup\limits_{s>0} \|\chi_{(0,st]}\|_{q,\mu} \|y^{-1}\chi_{[s,\infty)}(y)\|_{p',\sigma} & \text{if } p \le q, \\ \left(\int_0^\infty \left[\|\chi_{(0,st]}\|_{q,\mu} \|y^{-1}\chi_{[s,\infty)}(y)\|_{p',\sigma}^{p'/q'}\right]^r s^{-p'}\, d\sigma(s)\right)^{1/r} & \text{if } q < p. \end{cases}$$

If $X = R^n$, Q denotes a cube in R^n with centre x_Q and sides parallel to the coordinate axis of length $\ell(Q)$. If $j = (j_1, \dots, j_n)$ for integers $j_1, \dots, j_n$, let Q_j denote the translate of Q by $\ell(Q)j$ so that $x_{Q_j} = x_Q + \ell(Q)j$ and $\ell(Q_j) = \ell(Q)$. Let Q^* denote the dilate of Q by the factor 3 so that $x_{Q^*} = x_Q$ and $\ell(Q^*) = 3\ell(Q)$. Set $Q_j^* = (Q_j)^*$. Then

$$\omega(t) = \inf_{\{Q : \ell(Q) = t\}} \omega_Q$$

where

$$\omega_Q = \begin{cases} \sup\limits_j \|\chi_{Q_j}\|_{q,\mu} \|\chi_{Q_j^*}\|_{p',\sigma} & \text{if } p \le q, \\ \left(\sum\limits_j \left[\|\chi_{Q_j}\|_{q,\mu} \|\chi_{Q_j^*}\|_{p',\sigma}\right]^r\right)^{1/r} & \text{if } q < p. \end{cases}$$

Throughout, we adopt the usual convention that products of the form $0 \cdot \infty$ are taken to be zero and when $p = 1$ or $p = \infty$, expressions involving σ are interpreted as appropriate limits; for example, $\|y^{-1}\chi_{[s,\infty)}(y)\|_{p',\sigma}$ is taken to be $\operatorname{ess.sup}_{y \in [s,\infty)} [yv(y)]^{-1}$ when $p = 1$. Note that $w(t)$ is nonnegative, nondecreasing on R^+ and is continuous wherever it is finite. In particular, $\omega(\infty) = \lim_{t\to\infty} \omega(t)$ exists as an extended real number.

If $k(t)$ is a nonnegative, nonincreasing, right continuous function on R^+, set $k(\infty) = \lim_{t\to\infty} k(t)$ and denote by Λ_k the positive Borel measure on R^+ generated by k defined by $\Lambda_k(a, b] = k(a) - k(b)$; see [10, Chapter 11].

If $k(t) = \chi_{(0,1)}(t)$ the operator given by (1.1) with $X = R^+$ is the 'dual' Hardy operator $A_1 f(x) = \int_x^\infty f(y)\, dy/y$ for which $\omega(1) < \infty$ is a known (see [8], [1], [3], [7]) necessary and sufficient condition for (1.2) to hold. Moreover, in this case the smallest constant C in (1.2) satisfies $c_{1,p,q}\omega(1) \le C \le c_{2,p,q}\omega(1)$ for constants $c_{1,p,q}$ and $c_{2,p,q}$. Our main result generalizes this as follows.

Theorem 1. *Let $k(t)$ be nonnegative, nonincreasing, and right continuous on R^+. Suppose $1 \le p, q \le \infty$ and that μ, ν are positive Borel measures on X*

with $\omega(t) = \omega(X; \mu, \nu; q, p; t)$ *satisfying*

$$K = \int_{R^+} \omega(t)\, d\Lambda_k(t) + k(\infty)\omega(\infty) < \infty. \tag{1.3}$$

Then there is a constant $c_{X,p,q}$ *such that* (1.2) *holds with* $C = c_{X,p,q}K$ *for all* $f \in L^p(X, \nu)$.

It should be noted that the integral in (1.3) is equivalent to the improper Riemann-Stieltjes integral $\int_0^\infty \omega(t)\, d[-k(t)]$ whenever k and ω have no common point of discontinuity; in particular, this is the case if k is continuous or if $\omega(t) < \infty$ for all t.

With $X = R^+$ and each of μ and ν of the form $d\mu/dx = x^\alpha$, inequalities of the form (1.2) for various choices of k have been given in Chapter 9 of [4], see especially Theorems 319, 341(2), 342(2), and 360. Although Theorem 1 generalizes many of these results, here we elaborate only the case $k(t) = e^{-t}$ which leads by a change of variable to new inequalities for the Laplace transform $\mathscr{L}$ since in that case $\mathscr{L}f(x) = Tg(x)$ where $g(y) = y^{-1}f(y^{-1})$.

Corollary 1. *If* $1 \le p, q \le \infty$, *and* μ, ν *are positive Borel measures on* R^+ *with* $K < \infty$ *where*

$$K = \begin{cases} \int_0^\infty e^{-t} \sup\limits_{s>0} \|\chi_{(0,t/s]}\|_{q,\mu} \|\chi_{(0,s]}\|_{p',\sigma}\, dt & \text{if } p \le q, \\ \int_0^\infty e^{-t} \left(\int_0^\infty \left[\|\chi_{(0,t/s]}\|_{q,\mu} \|\chi_{(0,s]}\|_{p',\sigma}^{p'/q'} \right]^r d\sigma(s) \right)^{1/r} dt & \text{if } q < p \end{cases} \tag{1.4}$$

then there is a constant $c_{p,q}$ *such that*

$$\|\mathscr{L}f\|_{q,\mu} \le C\|f\|_{p,\nu} \tag{1.5}$$

with $C = c_{p,q}K$ *for all* $f \in L^p(R^+, \nu)$.

Bloom [2] proved two theorems, each giving a sufficient condition and a different necessary condition for (1.5) to hold with $d\mu = u\,dx$, $d\nu = v\,dx$, and $1 < p, q < \infty$. In particular, his second theorem asserts that if B_δ is defined by

$$B_\delta = \begin{cases} \sup\limits_{s>0} (\mathscr{L}u(\delta s))^{1/q} \|\chi_{(0,s]}\|_{p',\sigma} & \text{if } p \le q, \\ \left(\int_0^\infty \left[(\mathscr{L}u(\delta s))^{1/q} \|\chi_{(0,s]}\|_{p',\sigma}^{p'/q'} \right]^r d\sigma(s) \right)^{1/r} & \text{if } q < p, \end{cases}$$

then $B_1 < \infty$ is sufficient while $B_q < \infty$ is necessary for (1.5) to hold. Since

$$\int_0^{t/s} u \le e^{\delta t} \int_0^{t/s} e^{-\delta s x} u(x)\, dx \le e^{\delta t} \mathscr{L}u(\delta s),$$

it follows immediately that $K < \infty$ in (1.4) if $B_\delta < \infty$ for some $\delta < q$. Thus Corollary 1 contains the sufficient condition and narrows the gap between the necessary and the sufficient conditions of Bloom's Theorem 2. Similarly, it is not difficult to show that Corollary 1 also contains the sufficient condition of his Theorem 1.

Among other applications, Theorem 1 may be used to obtain inequalities for the operators given by the Gauss-Weierstrass kernel $W_y(x) = (4\pi y)^{-n/2} e^{-|x|/4y}$ and the Poisson kernel $P_y(x) = \Gamma(\frac{n+1}{2}) y [\pi(y^2 + |x|^2)]^{-(n+1)/2}$ where $(x, y) \in R^{n+}$, for which inequalities of the form (1.2) have been widely studied. In particular, we have the following result for the Poisson operator.

Corollary 2. *If* $1 \le p, q \le \infty$, $y > 0$, *and* μ, ν *are positive Borel measures on* R^n *with* $\omega(t) = \omega(R^n; \mu, \nu; q, p; t)$ *satisfying*

$$\int_1^\infty \frac{\omega(t)}{t^{n+2}}\,dt < \infty, \tag{1.6}$$

then

$$\left\| \int_{R^n} P_y(x-z) f(z)\,dz \right\|_{q,\mu} \le C_y \|f\|_{p,\nu} \tag{1.7}$$

with

$$C_y = \frac{3^{n/p}(n+1)\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} y \int_0^\infty \frac{t\omega(t)}{(y^2+t^2)^{(n+3)/2}}\,dt. \tag{1.8}$$

A simple calculation shows that for $1 < p = q < \infty$ and $d\mu/dx = d\nu/dx = (1+|x|)^\alpha$,

$$\omega(t) \le c_{n,p,\alpha} \begin{cases} t^n(1+t)^{\max(0,\,-n/p'+\alpha/p)}, & \alpha \ge 0, \\ t^n(1+t)^{\max(0,\,-n/p-\alpha/p)}, & \alpha < 0, \end{cases}$$

and hence (1.7) holds in this case if $-n-p < \alpha < n(p-1)+p$. Moreover, this range of α is best possible for (1.7); the necessity of $-n-p < \alpha$ follows by taking $f(z) = |z|/\log|z|$ for large $|z|$ and $f(z) = 0$ otherwise. A duality argument shows the necessity of $\alpha < n(p-1)+p$.

Note that if $1 \le p \le q \le \infty$ and there is a constant K with

$$\|\chi_Q\|_{q,\mu} \|\chi_Q\|_{p',\sigma} \le K[\ell(Q)]^n \tag{1.9}$$

for all cubes Q, then $\omega(t) \le K(3t)^n$ so Corollary 2 shows that (1.7) holds with constant C independent of y; the case $n = 1 \le p = q < \infty$ was obtained by Muckenhoupt [9, Theorem 2].

The condition (1.9) with $n = 1 < p = q < \infty$ and $d\mu/dx = d\nu/dx = v$ is the well-known A_p condition which characterizes [6] the weight functions v for which the Hilbert transform

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_R \frac{f(t)}{x-t}\,dt, \qquad x \in R,$$

satisfies

$$\int_R |Hf|^p v \le C \int_R |f|^p v \tag{1.10}$$

for a constant C independent of f. Combining this with Corollary 2 we prove, in section 3, the following weighted inequalities for the Goldberg transform [5] given by

$$Gf(x) = \text{p.v.}\frac{1}{\pi}\int_R \frac{\sin(x-t)}{(x-t)^2} f(t)\,dt, \qquad x \in R.$$

Theorem 2. *If* $1 < p < \infty$ *and* $v = d\nu/dx$ *is a weight function on* R *with* $\omega(t) = \omega(R; \nu, \nu; p, p; t)$ *satisfying*

$$\frac{\omega(t)}{t} + \int_1^\infty \frac{\omega(s)}{s^3}\,ds \le K \tag{1.11}$$

for some constant K *and all* $0 < t < 1$, *then there is a constant* C *depending on* p *and* K *such that*

$$\int_R |Gf|^p v \le C \int_R |f|^p v. \tag{1.12}$$

for all $f \in L^p(R, \nu)$.

Theorem 1 is proved in section 2. Constants are denoted by c or C, with or without subscripts, but are not necessarily the same from line to line.

2. Proof of Theorem 1

We prove the case $X = R^+$ first.

For $t > 0$ let the Hardy operators A_t be given by

$$A_t f(x) = \int_{x/t}^{\infty} f(y)\,\frac{dy}{y}, \qquad x \in R^+.$$

As noted in the introduction, A_1 satisfies

$$\|A_1 f\|_{q,\mu} \le C\|f\|_{p,\nu}$$

with $C = c_{2,p,q}\omega(R^+; \mu, \nu; q, p; 1)$ and hence a change of variable shows

$$\|A_t f\|_{q,\mu} \le c_{2,p,q}\omega(R^+; \mu, \nu; q, p; t)\|f\|_{p,\nu}. \tag{2.1}$$

Suppose now that $f \ge 0$ and let T_1 be the operator associated with the kernel $k_1(t) = k(t) - k(\infty)$. Then $k_1(\infty) = 0$ and hence Fubini's Theorem shows

$$\begin{aligned} T_1 f(x) &= \int_0^{\infty} f(y) \int_{(x/y,\infty)} d\Lambda_k(t)\,\frac{dy}{y} \\ &= \int_{R^+} d\Lambda_k(t) \int_{x/t}^{\infty} f(y)\,\frac{dy}{y} \\ &= \int_{R^+} A_t f(x)\, d\Lambda_k(t). \end{aligned}$$

Minkowski's inequality for integrals now yields

$$\begin{aligned} \|T_1 f\|_{q,\mu} &\le \int_{R^+} \|A_t f\|_{q,\mu}\, d\Lambda_k(t) \\ &\le c_{2,p,q}\left(\int_{R^+} \omega(t)\, d\Lambda_k(t)\right)\|f\|_{p,\nu} \end{aligned} \tag{2.2}$$

in view of (2.1).

On the other hand, Hölder's inequality shows

$$\begin{aligned} \|(T - T_1)f\|_{q,\mu} &= k(\infty)\|\chi_{R^+}\|_{q,\mu} \int_{R^+} |f(y)|\,\frac{dy}{y} \\ &\le k(\infty)\|\chi_{R^+}\|_{q,\mu}\|y^{-1}\chi_{R^+}(y)\|_{p',\sigma}\|f\|_{p,\nu} \\ &= k(\infty)c_{p,q}\omega(\infty)\|f\|_{p,\nu} \end{aligned}$$

where $c_{p,q} = 1$ if $p \le q$ and $c_{p,q} = (r/p')^{1/r}$ otherwise. This combined with (2.2) completes the proof for the case $X = R^+$.

Turning to the case $X = R^n$, we first prove that for $t > 0$ the operator

$$A_t f(x) = \int_{|x-y|<t} f(y)\,dy$$

satisfies

$$\|A_t f\|_{q,\mu} \le 3^{n/p} C \|f\|_{p,\nu} \tag{2.3}$$

with $C = \omega(R^n; \mu, \nu; q, p; t)$.

To prove (2.3), fix a cube Q with $\ell(Q) = t$. Then for fixed j and $x \in Q_j$, we have $\{y : |x-y| < t\} \subset Q_j^*$ and therefore

$$\begin{aligned} |A_t f(x)| &\le \int_{Q_j^*} |f(y)|\,dy \\ &\le \|\chi_{Q_j^*}\|_{p',\sigma} \|\chi_{Q_j^*} f\|_{p,\nu} \end{aligned} \tag{2.4}$$

by Hölder's inequality. The case $q = \infty$ of (2.3) follows easily from (2.4) so we give the details only for $q < \infty$. Now, if $q < \infty$, then (2.4) shows

$$\begin{aligned} \|A_t f\|_{q,\mu} &\le \left(\sum_j \int_{Q_j} |A_t f|^q \,d\mu \right)^{1/q} \\ &\le \left(\sum_j \|\chi_{Q_j}\|_{q,\mu}^q \|\chi_{Q_j^*}\|_{p',\sigma}^q \|\chi_{Q_j^*} f\|_{p,\nu}^q \right)^{1/q} \\ &\le \begin{cases} \big(\sup_j \|\chi_{Q_j}\|_{q,\mu} \|\chi_{Q_j^*}\|_{p',\sigma}\big) \big(\sum_j \|\chi_{Q_j^*} f\|_{p,\nu}^q\big)^{1/q} & \text{if } p \le q, \\ \big(\sum_j [\|\chi_{Q_j}\|_{q,\mu} \|\chi_{Q_j^*}\|_{p',\sigma}]^r\big)^{1/r} \big(\sum_j \|\chi_{Q_j^*} f\|_{p,\nu}^p\big)^{1/p} & \text{if } q < p < \infty, \\ \big(\sum_j [\|\chi_{Q_j}\|_{q,\mu} \|\chi_{Q_j^*}\|_{p',\sigma}]^r\big)^{1/r} \big(\sup_j \|\chi_{Q_j^*} f\|_{p,\nu}\big) & \text{if } q < p = \infty \end{cases} \end{aligned}$$

where we used Hölder's inequality with exponents p/q, $(p/q)' = r/q$ on the sum in the cases $q < p$. We have $\left(\sum_j \|\chi_{Q_j^*} f\|_{p,\nu}^q\right)^{1/q} \le \left(\sum_j \|\chi_{Q_j^*} f\|_{p,\nu}^p\right)^{1/p}$ for $p \le q$, and thus, in any case, we obtain

$$\begin{aligned} \|A_t f\|_{q,\mu} &\le \omega_Q \begin{cases} \big(\sum_j \|\chi_{Q_j^*} f\|_{p,\nu}^p\big)^{1/p} & \text{if } p < \infty, \\ \sup_j \|\chi_{Q_j^*} f\|_{p,\nu} & \text{if } p = \infty \end{cases} \\ &= \omega_Q \begin{cases} \big(\int_{R^n} \sum_j \chi_{Q_j^*} |f|^p \,d\nu\big)^{1/p} & \text{if } p < \infty, \\ \|f\|_{p,\nu} & \text{if } p = \infty. \end{cases} \end{aligned}$$

Since $\sum_j \chi_{Q_j^*}(y) \le 3^n$ a.e., we obtain (2.3) upon taking the infimum over cubes Q with $\ell(Q) = t$.

Now suppose $f \ge 0$ and let T_1 be the operator associated with the kernel $k_1(t) = k(t) - k(\infty)$. Then Fubini's Theorem yields

$$\begin{aligned} T_1 f(x) &= \int_{R^n} f(y) \int_{(|x-y|,\infty)} d\Lambda_k(t)\,dy \\ &= \int_{R^+} A_t f(x)\,d\Lambda_k(t) \end{aligned}$$

and the remainder of the proof is analogous to that of the case $X = R^+$ using (2.3) in place of (2.1); $c_{R^n,p,q} = 3^{n/p}$ will suffice. The details are omitted.

3. Proof of Theorem 2

For each integer j, let χ_j and χ_j^* denote the characteristic functions of $I_j = (j-1, j]$ and $I_j^* = (j-2, j+1]$ respectively. Then since

$$\left|\frac{\sin t}{t^2} - \frac{1}{t}\right| \le c, \qquad |t| \le 2,$$

it follows that

$$\left|Gf(x) - \sum_j [H(f\chi_j^*)(x)]\chi_j(x)\right| \le cT|f|(x)$$

where

$$Tf(x) = \frac{1}{\pi}\int_R \frac{f(t)}{1+|x-t|^2}\,dt.$$

Now, let $v_j(x)$ have period 6 and be given by

$$v_j(x) = \begin{cases} v(x) & \text{if } x \in I_j^*, \\ v(2j+2-x) & \text{if } x \in I_{j+3}^*. \end{cases}$$

Since (1.11) implies $\omega(t)/t \le 4^3K$ for $t \le 3$, by considering separately those intervals Q with $\ell(Q) \le 3$ and those with $\ell(Q) > 3$ it follows that there is a constant c independent of j such that (1.9) holds with $n = 1$, $p = q$, $d\mu/dx = d\nu/dx = v_j$, and K replaced by cK. Hence, (1.10) shows there is a constant C depending only on p and K such that

$$\begin{aligned} \int_R \left|\sum_j [H(f\chi_j^*)(x)]\chi_j(x)\right|^p v(x)\,dx &= \sum_j \int_{I_j} |H(f\chi_j^*)(x)|^p v_j(x)\,dx \\ &\le \sum_j \int_R |H(f\chi_j^*)(x)|^p v_j(x)\,dx \\ &\le C\sum_j \int_{I_j^*} |f(x)|^p v_j(x)\,dx \\ &\le 3C\int_R |f(x)|^p v(x)\,dx. \end{aligned} \tag{3.1}$$

On the other hand, Corollary 2 yields

$$\int_R [T|f|(x)]^p v(x)\,dx \le C\int_R |f(x)|^p v(x)\,dx \tag{3.2}$$

in view of (1.11).

Combining (3.1) and (3.2) yields (1.12) and completes the proof of Theorem 2.

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