WEIGHTED INEQUALITIES FOR CONVOLUTIONS

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Abstract. For certain convolution operators $T$ on $\mathbb{R}^+$ or $\mathbb{R}^n$, sufficient conditions are given which ensure that $T$ is bounded between weighted Lebesgue spaces. The class of operators considered includes many of classical interest; in particular, new inequalities are obtained for the Laplace transform, the Poisson integral on $\mathbb{R}^n \times \mathbb{R}^+$, and Goldberg’s transform.

1. Introduction

In a recent paper Bloom [2] obtained weighted inequalities for the Laplace transform $L$ using known weighted inequalities for the Hardy operator. The purpose of this paper is twofold. First, we show that these weighted Hardy inequalities may be used in a different and, we think, more elementary way to obtain weighted inequalities for a class of operators on $\mathbb{R}^+ = (0, \infty)$. Our results will be seen to include and sharpen Bloom’s results for $L$. Secondly, we show that our approach admits a natural extension to a class of operators on $\mathbb{R}^n$; applications will be illustrated by deriving new inequalities for Goldberg’s transform on $\mathbb{R}$ and the Poisson integral on the half space $\mathbb{R}^{n+} = \mathbb{R}^n \times \mathbb{R}^+$.

If $X$ denotes $\mathbb{R}^+$ or $\mathbb{R}^n$ and $\mu$ is a positive Borel measure on $X$, denote by $L^p(X, \mu)$ the weighted Lebesgue space of measurable functions $f$ on $X$ for which $\|f\|_{p, \mu} < \infty$ where

$$\|f\|_{p, \mu} = \begin{cases} \left( \int_X |f|^p \, d\mu \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \mu \text{ ess. sup } |f(y)| & \text{if } p = \infty. \end{cases}$$

If $f(y)$ is expressed as a formula in $y$ it will sometimes be convenient to slightly abuse the notation $\|f\|_{p, \mu}$ by writing instead $\|f(y)\|_{p, \mu}$.

The operators considered here are convolutions with a suitable kernel $k$ and are given by

$$Tf(x) = \begin{cases} \int_{\mathbb{R}^+} k(x/y) f(y) \, dy/y & \text{if } X = \mathbb{R}^+, \\ \int_{\mathbb{R}^n} k(|x - y|) f(y) \, dy & \text{if } X = \mathbb{R}^n. \end{cases}$$

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1129
Given such $T$ and exponents $1 < p, q < \infty$, sufficient conditions to be satisfied by positive Borel measures $\mu$ and $\nu$ are given which ensure that

\[(1.2) \quad \|Tf\|_{q, \mu} \leq C \|f\|_{p, \nu}\]

for a constant $C$ independent of $f \in L^p(X, \nu)$. Since $\nu = \nu_a + \nu_s$ where $\nu_a$ is absolutely continuous and $\nu_s$ is singular with respect to Lebesgue measure on $X$, there is no loss of generality in taking $\nu = \nu_a$ since the left side of (1.2) is unchanged if $f$ is redefined to be zero on the support of $\nu_s$. For convenience we set $\nu = d\nu_a/dx$ and $d\sigma/dx = \nu^{-1/(p-1)}$. As usual $p'$ denotes the conjugate exponent of $p$ given by $1/p + 1/p' = 1$ and if $q < p$ the exponent $r$ is defined by $1/r = 1/q - 1/p$. We define a function $\omega(t) = \omega(X; \mu, \nu; q, p; t)$ for $t > 0$ as follows, $\chi_E$ denoting the characteristic function of $E$.

If $X = R^+$,

$$\omega(t) = \left\{ \begin{array}{ll}
\sup_{s > 0} \|\chi_{[0,s]}\|_{q, \mu} \|y^{-1}\chi_{[s, \infty)}(y)\|_{p', \sigma} & \text{if } p \leq q, \\
\left( \int_0^\infty \|\chi_{[0,s]}\|_{q, \mu} \|y^{-1}\chi_{[s, \infty)}(y)\|_{p', \sigma} \right)^{r - p'} ds \right)^{1/r} & \text{if } q < p.
\end{array} \right.$$  

If $X = R^n$, $Q$ denotes a cube in $R^n$ with centre $x_Q$ and sides parallel to the coordinate axis of length $\ell(Q)$. If $j = (j_1, \ldots, j_n)$ for integers $j_1, \ldots, j_n$, let $Q_j$ denote the translate of $Q$ by $\ell(Q)j$ so that $x_{Q_j} = x_Q + \ell(Q)j$ and $\ell(Q_j) = \ell(Q)$. Let $Q^*$ denote the dilate of $Q$ by the factor 3 so that $x_{Q^*} = x_Q$ and $\ell(Q^*) = 3\ell(Q)$. Set $Q_j^* = (Q_j)^*$. Then

$$\omega(t) = \inf_{\{Q : \ell(Q) = t\}} \omega_Q$$

where

$$\omega_Q = \left\{ \begin{array}{ll}
\sup_j \|\chi_{Q_j}\|_{q, \mu} \|\chi_{Q_j^*}\|_{p', \sigma} & \text{if } p \leq q, \\
\left( \sum_j \|\chi_{Q_j}\|_{q, \mu} \|\chi_{Q_j^*}\|_{p', \sigma} \right)^{1/r} & \text{if } q < p.
\end{array} \right.$$  

Throughout, we adopt the usual convention that products of the form $0 \cdot \infty$ are taken to be zero and when $p = 1$ or $p = \infty$, expressions involving $\sigma$ are interpreted as appropriate limits; for example, $\|y^{-1}\chi_{[s, \infty)}(y)\|_{p', \sigma}$ is taken to be $\text{ess.sup}_{y \in [s, \infty)} [yv(y)]^{-1}$ when $p = 1$. Note that $w(t)$ is nonnegative, nondecreasing on $R^+$ and is continuous wherever it is finite. In particular, $\omega(\infty) = \lim_{t \to \infty} \omega(t)$ exists as an extended real number.

If $k(t)$ is a nonnegative, nonincreasing, right continuous function on $R^+$, set $k(\infty) = \lim_{t \to \infty} k(t)$ and denote by $\Lambda_k$ the positive Borel measure on $R^+$ generated by $k$ defined by $\Lambda_k(a, b] = k(a) - k(b)$; see [10, Chapter 11].

If $k(t) = \chi_{[0,1]}(t)$ the operator given by (1.1) with $X = R^+$ is the ‘dual’ Hardy operator $A_{1f}(x) = \int_x^\infty f(y) dy/y$ for which $\omega(1) < \infty$ is a known (see [8], [1], [3], [7]) necessary and sufficient condition for (1.2) to hold. Moreover, in this case the smallest constant $C$ in (1.2) satisfies $c_{1,p,q} \omega(1) \leq C \leq c_{2,p,q} \omega(1)$ for constants $c_{1,p,q}$ and $c_{2,p,q}$. Our main result generalizes this as follows.

**Theorem 1.** Let $k(t)$ be nonnegative, nonincreasing, and right continuous on $R^+$. Suppose $1 \leq p, q \leq \infty$ and that $\mu$, $\nu$ are positive Borel measures on $X$.
with \( \omega(t) = \omega(X; \mu, \nu; q, p; t) \) satisfying

\[
K = \int_{R^+} \omega(t) \, d\Lambda_k(t) + k(\infty)\omega(\infty) < \infty.
\]

Then there is a constant \( c_{X, p, q} \) such that (1.2) holds with \( C = c_{X, p, q}K \) for all \( f \in L^p(X, \nu) \).

It should be noted that the integral in (1.3) is equivalent to the improper Riemann-Stieltjes integral \( \int_0^\infty \omega(t) \, d[-k(t)] \) whenever \( k \) and \( \omega \) have no common point of discontinuity; in particular, this is the case if \( k \) is continuous or if \( \omega(t) < \infty \) for all \( t \).

With \( X = R^+ \) and each of \( \mu \) and \( \nu \) of the form \( d\mu/dx = x^\alpha \), inequalities of the form (1.2) for various choices of \( k \) have been given in Chapter 9 of [4], see especially Theorems 319, 341(2), 342(2), and 360. Although Theorem 1 generalizes many of these results, here we elaborate only the case \( k(t) = e^{-t} \) which leads by a change of variable to new inequalities for the Laplace transform \( \mathcal{L} \) since in that case \( \mathcal{L}f(x) = Tg(x) \) where \( g(y) = y^{-1}f(y^{-1}) \).

**Corollary 1.** If \( 1 \leq p, q \leq \infty \), and \( \mu, \nu \) are positive Borel measures on \( R^+ \) with \( K < \infty \) where

\[
(1.4) \quad K = \left\{ \begin{array}{ll}
\int_0^\infty e^{-t} \sup_{s>0} \|\chi_{(0, t/s]}\|_{p, \sigma} \, d\sigma(s) & \text{if } p \leq q, \\
\int_0^\infty e^{-t} \left( \int_0^s \|\chi_{(0, t/s]}\|_{p, \sigma} \right) \, d\sigma(s) & \text{if } q < p
\end{array} \right.
\]

then there is a constant \( c_{p, q} \) such that

\[
(1.5) \quad \|\mathcal{L}f\|_{q, \mu} \leq C\|f\|_{p, \nu}
\]

with \( C = c_{p, q}K \) for all \( f \in L^p(R^+, \nu) \).

Bloom [2] proved two theorems, each giving a sufficient condition and a different necessary condition for (1.5) to hold with \( d\mu = udx \), \( d\nu = vdx \), and \( 1 < p, q < \infty \). In particular, his second theorem asserts that if \( B_\delta \) is defined by

\[
B_\delta = \left\{ \begin{array}{ll}
\sup_{s>0} (\mathcal{L}u(\delta s))^{1/q} \|\chi_{(0, s]}\|_{p', \sigma} & \text{if } p \leq q, \\
\left( \int_0^\infty (\mathcal{L}u(\delta s))^{1/q} \|\chi_{(0, s]}\|_{p', \sigma} \, d\sigma(s) \right)^{1/r} & \text{if } q < p
\end{array} \right.
\]

then \( B_1 < \infty \) is sufficient while \( B_q < \infty \) is necessary for (1.5) to hold. Since

\[
\int_0^t s \leq e^\delta t \int_0^t e^{-\delta s} u(x) \, dx \leq e^\delta t \mathcal{L}u(\delta s),
\]

it follows immediately that \( K < \infty \) in (1.4) if \( B_\delta < \infty \) for some \( \delta < q \). Thus Corollary 1 contains the sufficient condition and narrows the gap between the necessary and the sufficient conditions of Bloom’s Theorem 2. Similarly, it is not difficult to show that Corollary 1 also contains the sufficient condition of his Theorem 1.

Among other applications, Theorem 1 may be used to obtain inequalities for the operators given by the Gauss-Weierstrass kernel \( W_y(x) = (4\pi y)^{-n/2}e^{-|x|^2/4y} \) and the Poisson kernel \( P_y(x) = \Gamma(\frac{n}{2})y[\pi(y^2 + |x|^2)]^{-(n+1)/2} \) where \( (x, y) \in R^{n+} \), for which inequalities of the form (1.2) have been widely studied. In particular, we have the following result for the Poisson operator.
Corollary 2. If $1 \leq p, q \leq \infty$, $y > 0$, and $\mu, \nu$ are positive Borel measures on $\mathbb{R}^n$ with $\omega(t) = \omega(R^n; \mu, \nu; q, p; t)$ satisfying

$$\int_1^\infty \frac{\omega(t)}{t^{n+2}} dt < \infty,$$

then

$$\left\| \int_{\mathbb{R}^n} P_y(x-z)f(z)dz \right\|_{q, \mu} \leq C_y\|f\|_{p, \nu}$$

with

$$C_y = \frac{3^{n/p}(n+1)\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} y \int_0^\infty \frac{t\omega(t)}{(y^2 + t^2)^{(n+3)/2}} dt.$$  \hfill (1.8)

A simple calculation shows that for $1 < p = q < \infty$ and $d\mu/dx = d\nu/dx = (1 + |x|)^\alpha$,

$$\omega(t) \leq c_{n, p, \alpha}\left\{ \begin{array}{ll} t^n(1+t)^{\max(0, -n/p' + \alpha/p)}, & \alpha \geq 0, \\ t^n(1+t)^{\max(0, -n/p - \alpha/p)}, & \alpha < 0, \end{array} \right.$$ 

and hence (1.7) holds in this case if $-n - p < \alpha < n(p - 1) + p$. Moreover, this range of $\alpha$ is best possible for (1.7); the necessity of $-n - p < \alpha$ follows by taking $f(z) = |z|/\log|z|$ for large $|z|$ and $f(z) = 0$ otherwise. A duality argument shows the necessity of $\alpha < n(p - 1) + p$.

Note that if $1 \leq p < q < \infty$ and there is a constant $K$ with

$$||\chi_Q||_{q, \mu}||\chi_Q||_{p', \sigma} \leq K[\ell(Q)]^n$$

for all cubes $Q$, then $\omega(t) \leq K(3t)^n$ so Corollary 2 shows that (1.7) holds with constant $C$ independent of $y$; the case $n = 1 \leq p = q < \infty$ was obtained by Muckenhoupt [9, Theorem 2].

The condition (1.9) with $n = 1 < p = q < \infty$ and $d\mu/dx = d\nu/dx = v$ is the well-known $A_p$ condition which characterizes [6] the weight functions $v$ for which the Hilbert transform

$$Hf(x) = p.v.\frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt, \quad x \in \mathbb{R},$$

satisfies

$$\int_{\mathbb{R}} |Hf|^p v \leq C \int_{\mathbb{R}} |f|^p v$$

for a constant $C$ independent of $f$. Combining this with Corollary 2 we prove, in section 3, the following weighted inequalities for the Goldberg transform [5] given by

$$Gf(x) = p.v.\frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin(x-t)}{(x-t)^2} f(t) dt, \quad x \in \mathbb{R}.$$ 

Theorem 2. If $1 < p < \infty$ and $v = dv/dx$ is a weight function on $\mathbb{R}$ with $\omega(t) = \omega(R; \nu, \nu; p, p; t)$ satisfying

$$\frac{\omega(t)}{t} + \int_1^\infty \frac{\omega(s)}{s^3} ds \leq K$$

(1.11)
for some constant $K$ and all $0 < t < 1$, then there is a constant $C$ depending on $p$ and $K$ such that

$$
\int_R |Gf|^p v \leq C \int_R |f|^p v.
$$

for all $f \in L^p(R, v)$.

Theorem 1 is proved in section 2. Constants are denoted by $c$ or $C$, with or without subscripts, but are not necessarily the same from line to line.

2. Proof of Theorem 1

We prove the case $X = R^+$ first.

For $t > 0$ let the Hardy operators $A_t$ be given by

$$A_t f(x) = \int_0^\infty f(y) \frac{dy}{y}, \quad x \in R^+.$$

As noted in the introduction, $A_1$ satisfies

$$\|A_1 f\|_{q, \mu} \leq C \|f\|_{p, \nu}$$

with $C = c_{2, p, q} \omega(R^+; \mu, \nu; q, p; 1)$ and hence a change of variable shows

$$\|A_t f\|_{q, \mu} \leq c_{2, p, q} \omega(R^+; \mu, \nu; q, p; t) \|f\|_{p, \nu}. \quad (2.1)$$

Suppose now that $f \geq 0$ and let $T_1$ be the operator associated with the kernel $k_1(t) = k(t) - k(\infty)$. Then $k_1(\infty) = 0$ and hence Fubini's Theorem shows

$$T_1 f(x) = \int_0^\infty f(y) \int_{x/y, \infty} d\Lambda_k(t) \frac{dy}{y} = \int_{R^+} d\Lambda_k(t) \int_{x/t}^\infty f(y) \frac{dy}{y} = \int_{R^+} A_t f(x) d\Lambda_k(t).$$

Minkowski's inequality for integrals now yields

$$\|T_1 f\|_{q, \mu} \leq \int_{R^+} \|A_t f\|_{q, \mu} d\Lambda_k(t) \leq c_{2, p, q} \left( \int_{R^+} \omega(t) d\Lambda_k(t) \right) \|f\|_{p, \nu} \quad (2.2)$$

in view of (2.1).

On the other hand, Hölder's inequality shows

$$\|(T - T_1)f\|_{q, \mu} = k(\infty) \|\chi_{R^+} f\|_{q, \mu} \int_{R^+} |f(y)| \frac{dy}{y} \leq k(\infty) \|\chi_{R^+} f\|_{q, \mu} \|f\|_{p', \sigma} \|f\|_{p, \nu} = k(\infty) c_{p, q} \omega(\infty) \|f\|_{p, \nu}$$

where $c_{p, q} = 1$ if $p \leq q$ and $c_{p, q} = (r/p')^{1/r}$ otherwise. This combined with (2.2) completes the proof for the case $X = R^+$. 

Turning to the case $X = R^n$, we first prove that for $t > 0$ the operator

$$A_t f(x) = \int_{|x-y| < t} f(y) \, dy$$

satisfies

$$\|A_t f\|_{q, \mu} \leq 3^n/p C \|f\|_{p, \nu}$$

with $C = \omega(R^n; \mu, \nu; q, p; t)$.

To prove (2.3), fix a cube $Q$ with $\ell(Q) = t$. Then for fixed $j$ and $x \in Q_j$, we have $\{y : |x - y| < t\} \subset Q_j^*$ and therefore

$$|A_t f(x)| \leq \int_{Q_j^*} |f(y)| \, dy \leq \|\chi_{Q_j^*}\|_{p'}, a \|\chi_{Q_j^*} f\|_{p, \nu}$$

by Hölder's inequality. The case $q = \infty$ of (2.3) follows easily from (2.4) so we give the details only for $q < \infty$. Now, if $q < \infty$, then (2.4) shows

$$\|A_t f\|_{q, \mu} \leq \left( \sum_j \int_{Q_j} |A_t f|^q \, d\mu \right)^{1/q} \leq \left( \sum_j \|\chi_{Q_j}\|_{q, \mu} \|\chi_{Q_j^*}\|_{p', \sigma} \|\chi_{Q_j^*} f\|_{p, \nu}^q \right)^{1/q} \leq \left( \sup_j \|\chi_{Q_j}\|_{q, \mu} \|\chi_{Q_j^*}\|_{p', \sigma} \right) \left( \sum_j \|\chi_{Q_j^*} f\|_{p, \nu}^q \right)^{1/q}$$

where we used Hölder's inequality with exponents $p/q, \ (p/q)' = r/q$ on the sum in the cases $q < p$. We have $\left( \sum_j \|\chi_{Q_j^*} f\|_{p, \nu}^q \right)^{1/q} \leq \left( \sum_j \|\chi_{Q_j^*} f\|_{p, \nu}^p \right)^{1/p}$ for $p \leq q$, and thus, in any case, we obtain

$$\|A_t f\|_{q, \mu} \leq \omega_Q \left\{ \begin{array}{ll} \left( \sum_j \|\chi_{Q_j^*} f\|_{p, \nu}^q \right)^{1/p} & \text{if } p < \infty, \\ \sup_j \|\chi_{Q_j^*} f\|_{p, \nu} & \text{if } p = \infty \end{array} \right. = \omega_Q \left\{ \begin{array}{ll} \left( \int_{R^n} \sum_j \|\chi_{Q_j^*} |f|^p \, d\nu \right)^{1/p} & \text{if } p < \infty, \\ \|f\|_{p, \nu} & \text{if } p = \infty. \end{array} \right.$$

Since $\sum_j \chi_{Q_j^*}(y) \leq 3^n$ a.e., we obtain (2.3) upon taking the infimum over cubes $Q$ with $\ell(Q) = t$.

Now suppose $f \geq 0$ and let $T_1$ be the operator associated with the kernel $k_1(t) = k(t) - k(\infty)$. Then Fubini's Theorem yields

$$T_1 f(x) = \int_{R^n} f(y) \int_{|x-y| < t} dA_k(t) \, dy = \int_{R^n} A_t f(x) \, dA_k(t)$$
and the remainder of the proof is analogous to that of the case $X = R^+$ using (2.3) in place of (2.1); $c_{R^+, p, q} = 3^{n/p}$ will suffice. The details are omitted.

3. Proof of Theorem 2

For each integer $j$, let $\chi_j$ and $\chi_j^*$ denote the characteristic functions of $I_j = (j - 1, j]$ and $I_j^* = (j - 2, j + 1]$ respectively. Then since

$$\left| \frac{\sin t}{t^2} - \frac{1}{t} \right| \leq c, \quad |t| \leq 2,$$

it follows that

$$\left| Gf(x) - \sum_j [H(f\chi_j^*)(x)]\chi_j(x) \right| \leq cT|f|(x)$$

where

$$Tf(x) = \frac{1}{\pi} \int_{R} \frac{f(t)}{1 + |x - t|^2} dt.$$

Now, let $v_j(x)$ have period 6 and be given by

$$v_j(x) = \begin{cases} v(x) & \text{if } x \in I_j^*, \\ v(2j + 2 - x) & \text{if } x \in I_{j+3}^*. \end{cases}$$

Since (1.11) implies $\omega(t)/t \leq 4^3 K$ for $t \leq 3$, by considering separately those intervals $Q$ with $\ell(Q) \leq 3$ and those with $\ell(Q) > 3$ it follows that there is a constant $c$ independent of $j$ such that (1.9) holds with $n = 1$, $p = q$, $d\mu/dx = dv/dx = v_j$, and $K$ replaced by $cK$. Hence, (1.10) shows there is a constant $C$ depending only on $p$ and $K$ such that

$$\int_R \left| \sum_j [H(f\chi_j^*)(x)]\chi_j(x) \right|^p v(x) dx = \sum_j \int_{I_j} |H(f\chi_j^*)(x)|^p v_j(x) dx$$

$$\leq \sum_j \int_R |H(f\chi_j^*)(x)|^p v_j(x) dx$$

$$\leq C \sum_j \int_{I_j} |f(x)|^p v_j(x) dx$$

$$\leq 3C \int_R |f(x)|^p v(x) dx. \quad (3.1)$$

On the other hand, Corollary 2 yields

$$\int_R [T|f|(x)]^p v(x) dx \leq C \int_R |f(x)|^p v(x) dx \quad (3.2)$$
in view of (1.11).

Combining (3.1) and (3.2) yields (1.12) and completes the proof of Theorem 2.

References


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