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HOLOMORPHIC GERMS ON TSIRELSON'S SPACE

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Dedicated to the memory of Leopoldo Nachbin (1922–1993)

ABSTRACT. We show that if K is an arbitrary compact subset of the Banach space constructed by Tsirelson, then the space $\mathscr{H}(K)$ of all holomorphic germs on K, with its natural inductive limit topology, is totally reflexive.

INTRODUCTION

If K is a compact subset of a complex Fréchet space E, then the space of holomorphic germs on K is defined as the inductive limit $\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$, where (U_n) is any decreasing fundamental sequence of open neighborhoods of K and $\mathscr{H}^\infty(U_n)$ is the Banach space of all bounded holomorphic functions on U_n. In [6] Bierstedt and Meise proved that the inductive limit $\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$ is compact if and only if E is a Fréchet-Schwartz space, and in [7] they posed the problem of characterizing those Fréchet spaces for which the inductive limit $\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$ is weakly compact. Not only does this problem still remain open, but until now no example of a Fréchet space was known, outside Fréchet-Schwartz spaces, for which the inductive limit $\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$ is weakly compact.

In this paper we remedy this situation by proving that the inductive limit $\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$ is weakly compact for each compact subset K of the Banach space X constructed by Tsirelson in [20]. In particular, $\mathscr{H}(K)$ is totally reflexive for each compact subset K of X. Boyd [8] had previously proved that $\mathscr{H}(K)$ is reflexive for each balanced, compact subset K of X.

As an application of our methods we show that, for each open subset U of X, the space $\mathscr{H}(U)$ of all holomorphic functions on U, with the Nachbin topology τ_w, is semireflexive. In the case of balanced open sets this result is due to Alencar, Aron, and Dineen [2].

As a further application of our methods we show that the space $\mathscr{H}_b(X)$ of all entire functions of bounded type on X is a totally reflexive Fréchet space.

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We refer to the books of Dineen [11] or Mujica [16] for the terminology from infinite-dimensional complex analysis.

1. Spaces of holomorphic germs

Let $\mathscr{P}(E)$ denote the algebra of all continuous polynomials on E, let $\mathscr{P}_w(E)$ denote the subalgebra of all $P \in \mathscr{P}(E)$ which are weakly continuous on bounded sets, and let $\mathscr{P}_f(E)$ denote the subalgebra generated by the dual E' of E. We always have that $\mathscr{P}_f(E) \subset \mathscr{P}_w(E) \subset \mathscr{P}(E)$, and the relations among these algebras have been extensively studied by Aron and Prolla [4] and by Aron, Hervés, and Valdivia [3].

The main tool in the proof of the next theorem is the following result of Mujica [17, Theorem 2.1]: For each open subset U of a Banach space E there are a Banach space $G^\infty(U)$ and a mapping $\delta_U \in \mathscr{H}^\infty(U; G^\infty(U))$ with the following universal property: For each Banach space F and each mapping $f \in \mathscr{H}^\infty(U; F)$ there is a unique operator $T_f \in \mathscr{L}(G^\infty(U); F)$ such that $T_f \circ \delta_U = f$.

1.1. **Theorem.** *Let E be a reflexive Banach space such that $\mathscr{P}(E) = \mathscr{P}_w(E)$. Then the inductive limit $\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$ is weakly compact, and in particular $\mathscr{H}(K)$ is totally reflexive, for each compact subset K of E.*

Proof. Let $U_n = K + V_n$, where (V_n) is a sequence of open balls in E, centered at the origin, with $V_{n+1} \subset \frac{1}{3} V_n$ for every $n \in \mathbb{N}$. By the aforementioned result of Mujica, there is an operator $S_n \colon G^\infty(K + V_{n+1}) \to G^\infty(K + V_n)$ such that the following diagram is commutative:

$$\begin{array}{ccc} K + V_{n+1} & \hookrightarrow & K + V_n \\ \Big\downarrow{\scriptstyle \delta_{K+V_{n+1}}} & & \Big\downarrow{\scriptstyle \delta_{K+V_n}} \\ G^\infty(K + V_{n+1}) & \xrightarrow[S_n]{} & G^\infty(K + V_n) \end{array}$$

Since K is compact, for each $n \in \mathbb{N}$ there is a finite set $A_n \subset K$ such that $K \subset A_n + V_n$. Hence

$$K + V_{n+1} \subset \bigcup_{a \in A_{n+1}} (a + 2V_{n+1}) \subset \bigcup_{a \in A_{n+1}} (a + \tfrac{2}{3} V_n).$$

If $f \in \mathscr{H}^\infty(a + V_n; F)$, then it follows from the hypotheses on E that the restriction of f to the set $a + \frac{2}{3} V_n$ is continuous for the topologies $\sigma(E, E')$ and $\sigma(F, F')$. By applying this remark to the mapping δ_{K+V_n}, we conclude that the set $\delta_{K+V_n}(K + V_{n+1})$ is a relatively weakly compact subset of $G^\infty(K + V_n)$. Then [17, Proposition 3.4] guarantees that the operator S_n is weakly compact. Since the restriction mapping $R_n \colon \mathscr{H}^\infty(K + V_n) \to \mathscr{H}^\infty(K + V_{n+1})$ is nothing but the transpose of S_n, we conclude that R_n is weakly compact as well. That $\mathscr{H}(K)$ is totally reflexive follows then from results of Komatsu [13, Theorems 6 and 8].

1.2. **Examples.** (a) Let X be the reflexive Banach space constructed by Tsirelson [20]. Then it follows from the proofs of results of Alencar, Aron, and Dineen [2, Proposition 4 and Theorem 6] that $\mathscr{P}(X) = \mathscr{P}_w(X)$.

(b) By using the notion of spreading models, Farmer [12] has given other examples of reflexive Banach spaces E such that $\mathscr{P}(E) = \mathscr{P}_w(E)$.

2. Spaces of polynomials

Let $\mathscr{P}(^mE)$ denote the Banach space of all m-homogeneous members of $\mathscr{P}(E)$, let $\mathscr{P}_w(^mE) = \mathscr{P}_w(E) \cap \mathscr{P}(^mE)$, and let $\mathscr{P}_f(^mE) = \mathscr{P}_f(E) \cap \mathscr{P}(^mE)$. If $\mathscr{H}(K)$ is reflexive, then each $\mathscr{P}(^mE)$ is reflexive as well, and it follows from a result of Alencar [1, Theorem 7] that the sufficient conditions in Theorem 1.1 are also necessary within the class of Banach spaces with the approximation property. To improve this result we begin with the following observation, where τ_c denotes the compact-open topology.

2.1. **Proposition.** *If E is a Banach space, then the following conditions are equivalent*:

(a) *$\mathscr{P}_w(E)$ is τ_c-dense in $\mathscr{P}(E)$.*
(b) *$\mathscr{P}_w(^mE)$ is τ_c-dense in $\mathscr{P}(^mE)$ for every $m \in \mathbb{N}$.*

The implication (b) $\Rightarrow$ (a) is obvious, whereas the converse implication follows at once from the Cauchy integral formula (see [16, Corollary 7.3.]).

It is well known (see [16, Theorem 28.1]) that if E is a Banach space with the approximation property, then $\mathscr{P}_f(E)$ is τ_c-dense in $\mathscr{P}(E)$. Similarly we have the following result.

2.2. **Proposition.** *If E is a Banach space with the compact approximation property, then $\mathscr{P}_w(E)$ is τ_c-dense in $\mathscr{P}(E)$.*

Proof. Let $P \in \mathscr{P}(E)$, let K be a compact subset of E, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that $|P(y) - P(x)| < \varepsilon$ whenever $x \in K$ and $y \in E$, with $\|y - x\| < \delta$. Let $T \in \mathscr{L}(E; E)$ be a compact operator such that $\|Tx - x\| < \delta$ for all $x \in K$. Then $|P \circ T(x) - P(x)| < \varepsilon$ for all $x \in K$. Since T maps bounded, weakly convergent nets into norm convergent nets, we see that the polynomial $P \circ T$ is weakly continuous on bounded sets.

2.3. **Examples.** (a) Pisier [18, Theorem 3.2] has given an example of a Banach space E, without the approximation property, such that $E \otimes_\pi E = E \otimes_\varepsilon E$. Whence $\mathscr{P}_f(^2E)$ is τ_c-dense in $\mathscr{P}(^2E)$.

(b) Willis [22, Propositions 3 and 4] has given an example of a reflexive Banach space E which has the compact approximation property but does not have the approximation property. In particular, $\mathscr{P}_w(E)$ is τ_c-dense in $\mathscr{P}(E)$.

(c) As pointed out by Lindenstrauss and Tzafriri [14, p. 94], Davie [9] has given examples of reflexive Banach spaces without the compact approximation property.

However, we have been unable to answer the following questions.

2.4. **Questions.** Is there a Banach space E such that $\mathscr{P}_f(E)$ is not τ_c-dense in $\mathscr{P}(E)$? Is there a Banach space E such that $\mathscr{P}_w(E)$ is not τ_c-dense in $\mathscr{P}(E)$?

The main tool in the proof of the next theorem is the following result of Ryan [19] (see also [17, Theorem 2.4]): For each Banach space E and each $m \in \mathbb{N}$ there are a Banach space $Q(^mE)$ and a mapping $\delta_m \in \mathscr{P}(^mE; Q(^mE))$ with the following universal property: For each Banach space F and each polynomial $P \in \mathscr{P}(^mE; F)$ there is a unique operator $T_P \in \mathscr{L}(Q(^mE); F)$ such that $T_P \circ \delta_m = P$. Furthermore, $Q(^mE) = (\mathscr{P}(^mE), \tau_c)'$.

2.5. **Theorem.** *Let E be a Banach space, and let $m \in \mathbb{N}$. Then the following conditions are equivalent:*

(a) *E is reflexive and $\mathscr{P}(^mE) = \mathscr{P}_w(^mE)$.*

(b) *$\mathscr{P}(^mE)$ is reflexive and $\mathscr{P}_w(^mE)$ is τ_c-dense in $\mathscr{P}(^mE)$.*

Proof. (a) $\Rightarrow$ (b): It follows from (a) that the polynomial δ_m maps bounded subsets of E into relatively weakly compact subsets of $Q(^mE)$. Then [17, Proposition 3.4] guarantees that the identity mapping on $Q(^mE)$ is weakly compact, and hence $Q(^mE)$ is reflexive. Thus $\mathscr{P}(^mE) = Q(^mE)'$ is reflexive too.

(b) $\Rightarrow$ (a): Since $\mathscr{P}(^mE)$ is reflexive, we have that $\mathscr{P}(^mE)' = Q(^mE) = (\mathscr{P}(^mE), \tau_c)'$, and therefore

$$\mathscr{P}(^mE) = \overline{\mathscr{P}_w(^mE)}^{\tau_c} = \overline{\mathscr{P}_w(^mE)}^{\|\cdot\|} = \mathscr{P}_w(^mE).$$

That E is reflexive follows from the fact that E' is topologically isomorphic to a complemented subspace of $\mathscr{P}(^mE)$ (see [5, Proposition 5.3]). But it follows also from the following lemma, whose simple proof was shown to us by Mário Matos.

2.6. **Lemma.** *Let E be a Banach space, and let $m \in \mathbb{N}$. Then E' is topologically isomorphic to a closed subspace of $\mathscr{P}(^mE)$.*

Proof. Fix $\psi \in E'$ with $\|\psi\| = 1$, and let $T: E' \to \mathscr{P}(^mE)$ be defined by $T\varphi = \varphi\psi^{m-1}$ for every $\varphi \in E'$. Then it follows from the polarization formula (see [16, Theorem 2.2]) that $(m!/m^m)\|\varphi\| \le \|T\varphi\| \le \|\varphi\|$ for every $\varphi \in E'$.

3. Spaces of holomorphic functions

3.1. **Theorem.** *Let E be a reflexive Banach space such that $\mathscr{P}(E) = \mathscr{P}_w(E)$. Then the space $(\mathscr{H}(U), \tau_w)$ is semireflexive for each open subset U of E.*

Proof. Let K be a compact subset of U, and let (U_n) be a decreasing fundamental sequence of open neighborhoods of K. Following [15] let $\mathscr{H}^K(U)$ denote the image of the natural mapping $\mathscr{H}(U) \to \mathscr{H}(K)$, and let $\widetilde{\mathscr{H}}^K(U)$ denote the inductive limit

$$\widetilde{\mathscr{H}}^K(U) = \operatorname{ind} \overline{\mathscr{H}^K(U) \cap \mathscr{H}^\infty(U_n)}, \tag{3.1}$$

where $\overline{\mathscr{H}^K(U) \cap \mathscr{H}^\infty(U_n)}$ denotes the closure of $\mathscr{H}^K(U) \cap \mathscr{H}^\infty(U_n)$ in $\mathscr{H}^\infty(U_n)$, with the induced norm. Since the inductive limit

$$\mathscr{H}(K) = \operatorname{ind} \mathscr{H}^\infty(U_n)$$

is weakly compact, by Theorem 1.1, it is clear that the inductive limit (3.1) is weakly compact too. In particular, the space $\widetilde{\mathscr{H}}^K(U)$ is totally reflexive. Now by [15, Lemma 5.6] the space $(\mathscr{H}(U), \tau_w)$ is topologically isomorphic to the projective limit of the spaces $\widetilde{\mathscr{H}}^K(U)$, with $K \subset U$. Whence the space $(\mathscr{H}(U), \tau_w)$ is semireflexive.

4. Spaces of holomorphic functions of bounded type

If U is an open subset of a Banach space E, then $\mathscr{H}_b(U) = \operatorname{proj} \mathscr{H}^\infty(U_n)$, where

$$U_n = \{x \in U : \|x\| < n \text{ and } d_U(x) > \tfrac{1}{n}\}.$$

4.1. **Theorem.** *Let E be a reflexive Banach space such that $\mathscr{P}(E) = \mathscr{P}_w(E)$. Then the projective limit $\mathscr{H}_b(U) = \operatorname{proj} \mathscr{H}^\infty(U_n)$ is weakly compact, and in particular $\mathscr{H}_b(U)$ is totally reflexive, for each balanced open subset U of E.*

Proof. If U is balanced, then each U_n is balanced as well, and there is θ_n, $0 < \theta_n < 1$, such that $U_n \subset \theta_n U_{n+1}$. Then we can show, as in the proof of Theorem 1.1, that the natural mapping $S_n\colon G^\infty(U_n) \to G^\infty(U_{n+1})$ is weakly compact. Whence it follows that the restriction mapping $R_n\colon \mathscr{H}^\infty(U_{n+1}) \to \mathscr{H}^\infty(U_n)$ is weakly compact as well. That $\mathscr{H}_b(U)$ is totally reflexive follows then, either from results of Komatsu [13, Theorems 1 and 3] or else from results of Davis et al. [10, Corollary 1] and Valdivia [21, Theorem 4].

Since $\mathscr{P}(^mE)$ is topologically isomorphic to a closed subspace of $\mathscr{H}(K)$ (resp. $(\mathscr{H}(U), \tau_w)$, resp. $\mathscr{H}_b(U)$), Theorems 1.1, 2.5, 3.1, and 4.1 yield the following corollary.

4.2. **Corollary.** *Let E be a Banach space such that $\mathscr{P}_w(E)$ is τ_c-dense in $\mathscr{P}(E)$. Then the following conditions are equivalent:*

(a) *$\mathscr{H}(K)$ is totally reflexive for each (for some) compact subset K of E.*
(b) *$\mathscr{H}(K)$ is reflexive for each (for some) compact subset K of E.*
(c) *$(\mathscr{H}(U), \tau_w)$ is semi-reflexive for each (for some) open subset U of E.*
(d) *$\mathscr{H}_b(U)$ is totally reflexive for each (for some) balanced open subset U of E.*
(e) *$\mathscr{H}_b(U)$ is reflexive for each (for some) balanced open subset U of E.*
(f) *$\mathscr{P}(^mE)$ is reflexive for every $m \in \mathbb{N}$.*
(g) *E is reflexive and $\mathscr{P}(E) = \mathscr{P}_w(E)$.*

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