UPPER BOUNDS FOR THE DERIVATIVE OF EXPONENTIAL SUMS

PETER BORWEIN AND TAMÁS ERDÉLYI

(Communicated by Andrew Bruckner)

Abstract. The equality
\[ \sup_p \frac{|p'(a)|}{||p||_{[a,b]}} = \frac{2n^2}{b-a} \]

is shown, where the supremum is taken for all exponential sums \( p \) of the form
\[ p(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j \in \mathbb{R}, \]

with nonnegative exponents \( \lambda_j \). The inequalities
\[ ||p'||_{[a+\delta, b-\delta]} \leq 4(n + 2)^3 \delta^{-1/2} ||p||_{[a, b]} \]

and
\[ ||p'||_{[a+\delta, b-\delta]} \leq 4\sqrt{2}(n + 2)^3 \delta^{-3/2} ||p||_{L^2[a,b]} \]

are also proved for all exponential sums of the above form with arbitrary real exponents. These results improve inequalities of Lorentz and Schmidt and partially answer a question of Lorentz.

1. Introduction and notation

Let \( \Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}, \lambda_j \neq 0, j = 1, 2 \ldots, n; \)
\[ E(\Lambda_n) := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, a_j \in \mathbb{R} \right\}; \]
and
\[ E_n := \bigcup_{\Lambda_n} E(\Lambda_n) = \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, a_j, \lambda_j \in \mathbb{R} \right\}. \]

We will use the norms
\[ ||f||_{[a, b]} := \max_{x \in [a, b]} |f(x)| \]

Received by the editors March 10, 1993 and, in revised form, August 18, 1993.
1991 Mathematics Subject Classification. Primary 41A17.
and
\[ \|f\|_{L^2[a,b]} := \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2} \]
for functions \( f \in C[a, b] \).

Schmidt [3] proved that there is a constant \( c(n) \) depending on \( n \) so that
\[ \|p'(\cdot)\|_{[a+\delta, b-\delta]} \leq c(n) \delta^{-1} \|p\|_{[a,b]} \]
for every \( p \in E_n \) and \( \delta \in (0, (b - a)/2) \). Lorentz [2] improved Schmidt's result by showing that for every \( \alpha > \frac{1}{2} \) there is a constant \( c(\alpha) \) depending only on \( \alpha \) so that \( c(n) \) in the above inequality can be replaced by \( c(\alpha) n^{\alpha \log n} \), and he speculated that there may be an absolute constant \( c \) so that Schmidt's inequality holds with \( c(n) = cn \). \(^1\) Theorem 2 of this paper shows that Schmidt's inequality holds with \( c(n) = 4(n + 2)^3 \). Our first theorem establishes the sharp inequality
\[ |p'(a)| \leq \frac{2n^2}{b-a} \|p\|_{[a,b]} \]
for every \( p \in E_n \) with nonnegative exponents \( \lambda_j \).

2. New results

**Theorem 1.** We have
\[ \sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a} \]
for every \( a < b \), where the supremum is taken for all exponential sums \( p \in E_n \) with nonnegative exponents. The equality
\[ \sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)} \]
also holds for every \( 0 < a < b \), where the supremum is taken for all Müntz polynomials of the form
\[ p(x) = a_0 + \sum_{j=1}^n a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \lambda_j \geq 0. \]

**Theorem 2.** The inequalities
\[ \|p'(\cdot)\|_{[a+\delta, b-\delta]} \leq 4(n + 2)^3 \delta^{-1} \|p\|_{[a,b]} \]
and
\[ \|p'(\cdot)\|_{[a+\delta, b-\delta]} \leq 4\sqrt{2}(n + 2)^{3/2} \delta^{-3/2} \|p\|_{L^2[a,b]} \]
hold for every \( p \in E_n \) and \( \delta \in (0, (b - a)/2) \).

3. Proofs

To prove Theorem 1 we need some notation. If \( \Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \) is a set of positive real numbers, then the real span of
\[ \{1, x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}\}, \quad x \geq 0, \]
\(^1\) (Added in proof) We can now prove this with \( c = 2 \); the proof will appear elsewhere.
will be denoted by $M(\Lambda_n)$. It is well known that these are Chebyshev spaces on $[0, \infty)$ (see [1] for instance), so $M(\Lambda_n)$ possesses a unique Chebyshev "polynomial" $T_{\Lambda_n}$ on $[a, b]$, $0 < a < b$, with the properties

(i) $T_{\Lambda_n} \in M(\Lambda_n)$,
(ii) $\|T_{\Lambda_n}\|_{[a, b]} = 1$, and
(iii) there are $a = x_0 < x_1 < \cdots < x_n = b$ so that

$$T_{\Lambda_n}(x_j) = (-1)^{n-j}, \quad j = 0, 1, \ldots, n.$$ 

It is routine to prove (see [1] again) that $T_{\Lambda_n}$ has exactly $n$ distinct zeros on $(a, b)$,

$$\max_{0 \neq p \in M(\Lambda_n)} \frac{|p'(a)|}{\|p\|_{[a, b]}} = \frac{|T_{\Lambda_n}'(a)|}{\|T_{\Lambda_n}\|_{[a, b]}} = |T_{\Lambda_n}'(a)|,$$

and

$$\max_{0 \neq p \in M(\Lambda_n)} \frac{|p(0)|}{\|p\|_{[a, b]}} = \frac{|T_{\Lambda_n}(0)|}{\|T_{\Lambda_n}\|_{[a, b]}} = |T_{\Lambda_n}(0)|.$$

Lemma 3. Let

$$\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_1 < \gamma_2 < \cdots < \gamma_n\}$$

so that $0 < \lambda_j \leq \gamma_j$ for each $j = 1, 2, \ldots, n$. Then

$$|T_{\Lambda_n}'(a)| \leq |T_{\Lambda_n}'(a)|. \quad (3)$$

Proof. Without loss of generality we may assume that there is an index $m$, $1 \leq m \leq n$, so that $\lambda_m < \gamma_m$ and $\lambda_j = \gamma_j$ if $j \neq m$, since repeated applications of the result in this situation give the lemma in the general case. First we show that

$$|T_{\Gamma_n}(0)| < |T_{\Lambda_n}(0)|. \quad (4)$$

Indeed, let $R_{\Gamma_n} \in M(\Gamma_n)$ interpolate $T_{\Lambda_n}$ at the zeros of $T_{\Lambda_n}$ and be normalized so that $R_{\Gamma_n}(0) = T_{\Lambda_n}(0)$. Then the Improvement Theorem of Pinkus and Smith [4, Theorem 2] yields

$$|R_{\Gamma_n}(x)| \leq |T_{\Lambda_n}(x)| \leq 1, \quad x \in [a, b].$$

Hence, using (2) with $\Lambda_n$ replaced by $\Gamma_n$, we obtain

$$|T_{\Lambda_n}(0)| = |R_{\Gamma_n}(0)| \leq |T_{\Gamma_n}(0)|,$$

which proves (4). Using the defining properties of $T_{\Lambda_n}$ and $T_{\Gamma_n}$, we can deduce that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 1$ zeros in $[a, b]$ (we count every internal zero without sign change twice). Now assume that (3) does not hold; then

$$|T_{\Lambda_n}'(a)| > |T_{\Gamma_n}'(a)|.$$

This, together with (4), implies that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least one zero in $(0, a)$. Hence $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 2$ zeros in $(0, b]$. This is a contradiction, since

$$T_{\Lambda_n} - T_{\Gamma_n} \in \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}, x^{\gamma_m}\},$$

and every function from the above span can have only at most $n + 1$ zeros in $(0, \infty)$ (see [3]). \qed
Proof of Theorem 1. It is sufficient to prove only the second statement of the theorem, the first one can be obtained by the change of variable $x = e^t$. We obtain from (1) and Lemma 3 that

$$
\frac{|p'(a)|}{\|p\|_{[a,b]}} \leq \lim_{\delta \to 0^+} \frac{|T_{\Lambda_{n,\delta}}'(a)|}{\|T_{\Lambda_{n,\delta}}\|_{[a,b]}} = \lim_{\delta \to 0^+} \frac{|T_{\Lambda_{n,\delta}}'(a)|}{\delta}
$$

for every $p$ of the form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \lambda_j > 0,$$

where

$$\Lambda_{n,\delta} := \{\delta, 2\delta, 3\delta, \ldots, n\delta\}$$

and $T_{n,\delta}$ is the Chebyshev "polynomial" of $M(\Lambda_{n,\delta})$ on $[a,b]$. From the definition and uniqueness of $T_{\Lambda_{n,\delta}}$ it follows that

$$T_{n,\delta}(x) = T_n \left( \frac{2}{b^\delta - a^\delta} x^\delta - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right),$$

where $T_n(y) := \cos(n \arccos y)$. Therefore,

$$|T_{\Lambda_{n,\delta}}'(a)| = |T_n'(\arccos(-1))| \frac{2}{b^\delta - a^\delta} \frac{a^{\delta-1}}{\delta-1} \frac{2n^2}{a(\log b - \log a)}$$

and the theorem is proved. \(\square\)

To prove Theorem 2 we need two lemmas.

Lemma 4. For every set $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ of nonzero real numbers there is a point $y \in [-1, 1]$ depending only on $\Lambda_n$ so that

$$|p'(y)| \leq 2(n + 2)^3 \|p\|_{L_2[-1,1]}$$

for every $p \in E(\Lambda_n)$.

Proof. Take the orthonormal set $\{p_k\}_{k=0}^{n}$ on $[-1, 1]$ defined by

(i) $p_k \in \text{span}\{1, e^{\lambda_i t}, e^{\lambda_j t}, \ldots, e^{\lambda_k t}\}, \quad k = 0, 1, \ldots, n$;

(ii) $\int_{-1}^{1} p_ip_j = \delta_{i,j}, \quad 0 \leq i \leq j \leq n$.

Writing $p \in E(\Lambda_n)$ as a linear combination of the functions $p_k$, $k = 0, 1, \ldots, n$, and using the Cauchy-Schwartz inequality and the orthonormality of $\{p_k\}_{k=0}^{n}$ on $[-1, 1]$, we obtain in a standard fashion that

$$\max_{p \in E(\Lambda_n)} \frac{|p'(t_0)|}{\|p\|_{L_2[-1,1]}} = \left( \sum_{k=0}^{n} p_k'(t_0)^2 \right)^{1/2}, \quad t_0 \in \mathbb{R}.$$

Let

$$A_k := \{t \in [-1, 1]: |p_k(t)| \geq (n + 1)^{1/2}\}, \quad k = 0, 1, \ldots, n,$$

and

$$B_k := \{t \in [-1, 1] \setminus A_k: |p_k(t)| \geq 2(n + 2)^{5/2}\}, \quad k = 0, 1, \ldots, n.$$
Since \( \int_{-1}^{1} p_k^2 = 1 \), we have
\[
m(A_k) \leq (n + 1)^{-1}, \quad k = 0, 1, \ldots, n.
\]

Since \( \text{span}\{1, e^{\lambda_k t}, e^{2\lambda_k t}, \ldots, e^{k\lambda_k t}\} \) is a Chebyshev system, each \( A_k := [-1, 1] \backslash A_k \) comprises of at most \( k + 1 \) intervals and each \( B_k \) comprises of at most \( 2(k + 1) \) intervals. Therefore,
\[
2(n + 2)^{5/2} m(B_k) \leq \int_{B_k} |p_k'(t)| \, dt \leq 4(k + 1)\sqrt{n + 1},
\]
whence
\[
\sum_{k=0}^{n} m(B_k) \leq \frac{2\sqrt{n + 1} (n + 1)(n + 2)}{(n + 2)^{5/2}} < 1.
\]

Now let
\[
A := [-1, 1] \backslash \bigcup_{k=0}^{n} (A_k \cup B_k).
\]

Then
\[
m(A) \geq 2 - \sum_{k=0}^{n} m(A_k) - \sum_{k=0}^{n} m(B_k)
\]
\[
> 2 - (n + 1)(n + 1)^{-1} - 1 = 0,
\]
so there is a point \( y \in A \subseteq [-1, 1] \) where
\[
|p'(y)| \leq 2(n + 1)^{5/2}, \quad k = 0, 1, \ldots, n.
\]

Hence,
\[
\left( \sum_{k=0}^{n} p_k'(y)^2 \right)^{1/2} \leq 2(n + 2)^3,
\]
and the lemma is proved. \( \square \)

**Lemma 5.** We have
\[
|p'(0)| \leq 2(n + 2)^3 \|p\|_{L_2[-2, 2]} \leq 2(n + 2)^3 \|p\|_{L_2[-2, 2]}
\]
for every \( p \in E_n \).

**Proof.** Let \( \Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \) be a fixed set of nonzero real numbers, and let \( y \in [-1, 1] \) be chosen by Lemma 4. Let \( 0 \neq p \in E(\Lambda_n) \). Then
\[
q(t) := p(t - y) \in E(\Lambda_n);
\]
therefore, applying Lemma 4 to \( q \), we obtain
\[
\frac{|p'(0)|}{\|p\|_{L_2[-2, 2]}} \leq \frac{|p'(0)|}{\|p\|_{L_2[-1 - y, 1 - y]}} = \frac{|q'(y)|}{\|q\|_{L_2[-1, 1]}} \leq 2(n + 2)^3,
\]
and the lemma is proved. \( \square \)

**Proof of Theorem 2.** Let \( t_0 \in [a + \delta, b - \delta] \). Applying Lemma 5 to \( q(t) := p(\delta t/2 + t_0) \), we get the theorem. \( \square \)
REFERENCES


DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA V5A 1S6

E-mail address: PBORWEIN@CECM.SFU.CA

E-mail address: ERDELYI@CS.SFU.CA