ON A SEQUENCE TRANSFORMATION WITH INTEGRAL COEFFICIENTS FOR EULER'S CONSTANT

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ABSTRACT. Let \( \gamma \) denote Euler's constant, and let
\[
  s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2).
\]
We prove by Ser's formula for the remainder \( \gamma - s_n \) that for all integers \( n \geq 1 \) and \( \tau \geq 2 \) there are integers \( \mu_{n,0}, \mu_{n,1}, \ldots, \mu_{n,n} \) such that
\[
  \mu_{n,0}s_1 + \mu_{n,1}s_1 + \cdots + \mu_{n,n}s_{n+n} = \gamma + O((n(n+1)(n+2)\cdots(n+\tau))^{-1})
\]
where the constant in \( O \) depends only on \( \tau \).

The coefficients \( \mu_{n,k} \) are explicitly given and are bounded by \( 2^{3n+\tau-1} \).

By \( \gamma \) we denote Euler's constant; it is well known that the sequence \( (s_n)_{n \geq 0} \) defined by
\[
  s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2)
\]
tends to \( \gamma \), where
\[
  s_n = \gamma + O(n^{-1}) \quad (n \geq 2).
\]
J. Ser [6] has proved that the remainder of \( \gamma - s_n \) \( (n \geq 2) \) can be expressed as an infinite sum with rational terms: Let
\[
  t_{m+2} = -\frac{1}{(m+1)!} \int_0^1 (1-x)^m \cdot (m-x) \, dx \quad (m \geq 0).
\]
Then
\[
  \gamma = \frac{1}{n} \sum_{m=0}^{\infty} \frac{t_{m+2}}{(m+n)^m} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2).
\]
(See also [3, pp. 14–15].)

But, of course, \( \gamma - s_n \) can be written in a lot of different ways. For example, we get by Euler's summation formula for any positive integers \( n \geq 2 \) and \( k \):
\[
  \gamma = s_n + \frac{1}{2n} + \sum_{j=1}^{k} \frac{B_{2j}}{2j \cdot n^{2j}} + R(n,k),
\]

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where $B_m$ are the Bernoulli numbers and

$$|R(n, k)| \leq \frac{4}{n \sqrt{\pi}} \left( \frac{k}{n\pi} \right)^{2k}$$

(see [4]).

**A historical remark.** The representation of $\gamma$ by the right-hand side of (2) was the main tool in P. Appell's attempt to prove the irrationality of $\gamma$ in 1926 [1]. Appell himself quickly discovered his error and within a week he published a retraction. An outline of this incorrect proof is sketched in [2]. In what follows we apply a linear sequence transformation to the class of those sequences, where the error term can be expressed by a sum like (2). First we introduce some notation:

$$(\alpha)_m = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + m - 1), \quad (\alpha)_0 = 1 \quad (\alpha \in \mathbb{R}, \ m \in \mathbb{Z}_{\geq 0});$$

$$\mu_{n, k}(\tau) = \mu_{n, k} = (-1)^{n+k} \frac{(\tau + k)n}{n!} \binom{n}{k} \quad (n \in \mathbb{Z}_{\geq 0}, \ 0 \leq k \leq n),$$

where $\tau \in \mathbb{Z}_{>0}$ is fixed. Note that $\mu_{n, k} \in \mathbb{Z}$ ($n \in \mathbb{Z}_{\geq 0}, \ 0 \leq k \leq n$).

**Theorem 1.** Let $(v_n)_{n \geq 0}$ be a sequence of real numbers such that

$$\lim_{n \to \infty} v_n = s,$$

$$(3) \quad v_n = s - \sum_{m=1}^{\infty} \frac{c_m}{(n + \tau)_m} \quad (n \geq 0),$$

where $(c_m)_{m \geq 1}$ denotes a sequence of real numbers satisfying

$$(4) \quad 0 \leq c_m \leq C \cdot (m + \sigma)! \quad (m \geq \max\{1; -\sigma\})$$

for some constant $C > 0$ and some

$$(5) \quad \sigma \in \mathbb{Z} \text{ with } \sigma < \tau - 2.$$

Then we have for

$$(6) \quad e_n = \left( \sum_{k=0}^{n} \mu_{n, k} v_k \right) - s :$$

$$|e_n| \leq C \cdot \frac{(n + \sigma + 1)! \cdot (\tau - \sigma - 3)!}{(n + \tau - \sigma - 2)!} \quad (n \geq \max\{0; -(\sigma + 1)\}).$$

The linear sequence transformation given in (6) belongs to a certain class of so-called nonregular methods; a general theory of such transformations can be found in [7] (see Chapter 2.3.5).

**Theorem 2.** For $n \geq 1$ and $\tau \geq 2$ we have

$$\left| \sum_{k=0}^{n} \mu_{n, k} s_{k+\tau} - \gamma \right| \leq \frac{(\tau - 1)!}{2n(n+1)(n+2) \cdots (n+\tau)}.$$

From this theorem we get a very good approximation to $\gamma$ in terms of $s_n, s_{n+1}, \ldots, s_{2n}$ by choosing $\tau = n \geq 2$:

$$\left| \sum_{k=0}^{n} \mu_{n, k} s_{n+k} - \gamma \right| \leq \frac{1}{2n^2(\binom{2n}{n})} \leq n^{-3/2} \cdot 4^{-n}.$$
There are linear sequence transformations for \((s_n)_n \geq 0\) with nonintegral coefficients, which converge more rapidly to \(x\) than the transformation given in Theorem 2 (see [5]). But from an arithmetical point of view in number theory it is much more attractive to accelerate the convergence by transformations with integral coefficients.

**Proof of the theorems.** From \(\sum_{k=0}^{n} \mu_{n,k} = 1\) we have by (3) and (6) for every \(n \geq 0\):

\[ e_n = -\sum_{k=0}^{n} \sum_{m=1}^{\infty} \frac{(-1)^{n+k}}{k! \cdot (n-k)! \cdot (k+m)} c_m \]

\[ = -\sum_{m=1}^{\infty} c_m \sum_{k=0}^{n} \frac{(-1)^{n+k}}{k! \cdot (n-k)! \cdot (k+m)} \]

From \(c_m \geq 0\) in (4) we conclude that the infinite series \(\sum_{m=1}^{\infty} c_m \frac{(-1)^{n+k}}{k! \cdot (n-k)! \cdot (k+m)}\) converges absolutely, and so we may interchange the sums in (7). We express the terms in (8) again by Pochhammer's symbol; this gives for \(n \geq 0\):

\[ e_n = (-1)^{n+1} \frac{(n+\tau-1)!}{n!} \sum_{m=1}^{\infty} c_m \sum_{k=0}^{n} \frac{(n+\tau)_k \cdot (-n)_k}{(m+\tau-1)! \cdot (k+m)} \]

\[ = (-1)^{n+1} \frac{(n+\tau-1)!}{n!} \cdot \left(\sum_{m=1}^{n} \frac{c_m}{(m+\tau-1)! \cdot (m+\tau)} \sum_{k=0}^{\infty} \frac{(n+\tau)_k \cdot (-n)_k}{(m+\tau-1)! \cdot (k+m)}\right) \]

(since \((-n)_k = 0\) if \(k > n\)). Let \(a, b, c\) be real numbers, \(c \neq 0, -1, -2, \ldots\);

\[ F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k \cdot (b)_k}{(c)_k} x^k. \]

We only treat the case \(c - a - b > 0\); for this it is well known that

\[ F(a, b; c; 1) = \begin{cases} \frac{\Gamma(c) \cdot \Gamma(c-a-b)}{\Gamma(c-a) \cdot \Gamma(c-b)} & \text{if } c - a, c - b \neq 0, -1, -2, \ldots, \\ 0 & \text{otherwise}. \end{cases} \]

The sum on the right-hand side of (10) occurs in (9) with

\[ a = n + \tau, \quad b = -n, \quad c = m + \tau. \]

From \(m \geq 1\) in (9) we have \(m + \tau > \tau\), hence \(c > a + b\). Note \(c - a \leq 0 \Rightarrow m \leq n\). By (11) we now see that \(e_n\) equals

\[ (-1)^{n+1} \frac{(n+\tau-1)!}{n!} \sum_{m=n+1}^{\infty} \frac{c_m}{(m+\tau-1)! \cdot (m+n)} \]

\[ = (-1)^{n+1} \frac{(n+\tau-1)!}{n!} \sum_{m=0}^{\infty} c_{m+n+1} \frac{(m+n)!}{m! \cdot (m+2n+\tau)} \quad (n \geq 0). \]

Now let

\[ n_0 = \max\{0; -(\sigma + 1)\}. \]
$n \geq n_0$ implies $m+n+1 \geq n+1 \geq \max\{1,-\sigma\}$. Thus for $n \geq n_0$ we estimate $e_n$ from (12) by (4),

$$|e_n| \leq C \cdot \frac{(n+\tau-1)!}{n!} \sum_{m=0}^{\infty} \frac{(m+n+\sigma+1)! \cdot (m+n)!}{m! \cdot (m+2n+\tau)!} \quad (n \geq n_0).$$

We treat the infinite sum in (13) in the same way as we did with the inner sum in (8). For $n \geq n_0$ we get

$$|e_n| \leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)!}{(2n+\tau)!} \sum_{m=0}^{\infty} \frac{(n+\sigma+2)_m \cdot (n+1)_m}{m! \cdot (2n+\tau+1)_m}. \quad (14)$$

To apply (11) again we now define in (10):

$$a = n + \sigma + 2, \quad b = n + 1, \quad c = 2n + \tau + 1.$$ 

From (5) we have $2n+\tau+1 > 2n + \sigma + 3$, hence $c > a + b$. That gives

$$|e_n| \leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)! \cdot (2n+\tau+1) \cdot \Gamma(\tau-\sigma-2)}{\Gamma(n+\tau-\sigma-1) \cdot \Gamma(n+\tau)}$$

$$= C \cdot \frac{(n+\sigma+1)! \cdot (\tau-\sigma-3)!}{(n+\tau-\sigma-2)!} \quad (n \geq n_0).$$

This proves the theorem.

Theorem 2 follows immediately from Theorem 1 and (2): Put

$$c_m = -\frac{1}{m} \int_0^1 (0-x)(1-x) \cdot \ldots \cdot (m-1-x) \, dx \quad (m \geq 1). \quad (15)$$

Hence

$$t_{m+1} = \frac{1}{(m-1)!} \cdot c_m \quad (m \geq 1);$$

from the definition of $s_n$ and (2) we get\(^1\)

$$s_{n+\tau} = \gamma - \frac{1}{n+\tau} \sum_{m=1}^{\infty} \frac{t_{m+1}}{(m+n+\tau-1)_m} = \gamma - \sum_{m=1}^{\infty} c_m \cdot \frac{(n+\tau-1)!}{(m+n+\tau-1)!}$$

$$= \gamma - \sum_{m=1}^{\infty} \frac{c_m}{(n+\tau)_m} \quad (n \geq 0).$$

This is (3), where $\tau \geq \mathbb{Z}_{\geq 2}$. We get an integer $\sigma$ from (15) by

$$0 \leq c_m \leq \frac{1}{m} \int_0^1 x \cdot (m-1)! \, dx = \frac{(m-1)!}{2m} \leq \frac{(m-2)!}{2} \quad (m \geq 2).$$

Hence we may choose $\sigma = -2$, $C = \frac{1}{2}$, $n_0 = 1$; and (5) holds.

Theorem 2 now follows from Theorem 1, where $v_n = s_{n+\tau}$. At last note that

$$\mu_{n,k} = (-1)^{n+k} \frac{(\tau+k)n}{n!} \binom{n}{k} = (-1)^{n+k} \binom{n+k+\tau-1}{n} \binom{n}{k}$$

$$= (-1)^{n+k} \binom{n+k+\tau-1}{n-k, k, k+\tau-1} \quad (n \geq 1, \ 0 \leq k \leq n)$$

\(^1\)Note that (2) holds for $s_{n+\tau}$ with $n \geq 0$ and $\tau \geq 2$. 

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and
\[ | \mu_{n,k} | \leq 2^{n+k+r-1} \cdot 2^{n} \leq 2^{3n+r-1}. \]

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**References**