ON THE COVERING AND THE ADDITIVITY NUMBER OF THE REAL LINE

KYRIAKOS KEREMEDIS

(Communicated by Andreas R. Blass)

Abstract. We show that the real line \( R \) cannot be covered by \( k \) many nowhere dense sets iff whenever \( D = \{ D_i : i \in k \} \) is a family of dense open sets of \( R \) there exists a countable dense set \( G \) of \( R \) such that \( |G \setminus D_i| < \omega \) for all \( i \in k \). We also show that the union of \( k \) meagre sets of the real line is a meagre set iff for every family \( D = \{ D_i : i \in k \} \) of dense open sets of \( R \) and for every countable dense set \( G \) of \( R \) there exists a dense set \( Q \subseteq G \) such that \( |Q \setminus D_i| < \omega \) for all \( i \in k \).

1. Notation and terminology

The notation and terminology which we will use is standard and can be found in [8] or [9]. In particular, if \( A, B \) are sets and \( \lambda \) any cardinal finite or infinite, then \([A]^{\lambda}, [A]^{< \lambda}, \text{and } [A]^{\leq \lambda}\) denote the sets of all subsets of \( A \) of size \( \lambda < \omega \), and \( \leq \lambda \) respectively. \( \omega^\omega \) denotes the Baire space, i.e., the set of all functions \( \omega^\omega \) together with the topology, having as a base the collection of all (clopen) sets of the form

\[
[p] = \{ f \in \omega^\omega : p \subseteq f \}, \quad p \in \omega^{< \omega} = \bigcup\{\{n \in \omega : n \in \omega\}.
\]

We say that the family \( A \subseteq [\omega]^\omega \) has the countable \( \pi \)-base property (c\( \pi \)bp), iff there exists a \( B \in ([\omega]^\omega)^\omega \), call it a \( \pi \)-base, such that

\[
(\forall a \in A)(\forall b \in B)(\exists d \in B)(d \subseteq a \cap b).
\]

In order to avoid confusion, let us remark that the notions of c\( \pi \)bp and a filter \( A \) of \([\omega]^\omega \) having a countable base are not the same. \( B \) need not be included in \( A \). In fact, \( B \) may contain disjoint sets.

\( A \subseteq [\omega]^\omega \) has the strong finite intersection property (sfip), iff \( |\bigcap Q| = \omega \) for every \( Q \in [A]^\omega \).

We say that a set \( S \in [\omega]^\omega \) is an infinite pseudointersection of the family \( A \subseteq [\omega]^\omega \) iff \( (\forall a \in A)(|S \setminus a| < \omega) \).

Received by the editors April 1, 1993 and, in revised form, September 10, 1993; presented to the 3rd Greek Conference on Mathematical Analysis held in Ioannina, May 28–29, 1993.

1991 Mathematics Subject Classification. Primary 03E35; Secondary 03E40.

Key words and phrases. Covering number, additivity number, nowhere dense, meagre, bounding number, dominating number.
Let \( k, m \) be any infinite cardinal numbers less than the power of the continuum \( c \). We define the following combinatorial statements:

**PS\(_1\)(\( k \)) =** Every \( A \in [[\omega]^\omega]^{\leq k} \) with the \( \pi \)-bp has an infinite pseudointersection.

We remark here that it is consistent with ZFC that there exist sets \( A \subseteq [\omega]^\omega \) having the sfip but no countable \( \pi \)-base (there are filters having no countable base). Indeed, let \( (\mathcal{M}, \epsilon) \) be a model of ZFC such that \( \mathcal{M} \models p = \omega_1 + \text{cov}(\mathcal{R}) = \omega_2 \) (see [9, Theorem 3]), where \( p \) is the pseudointersection number and \( \text{cov}(\mathcal{R}) \) the covering number of the real line. Then there exists a set \( A \subseteq [\omega]^\omega \) of cardinality \( \omega_1 \), having the sfip but no infinite pseudointersection. If \( B = \{ b_n : n \in \omega \} \subseteq [\omega]^\omega \) were a \( \pi \)-base for \( A \), then, using \( \text{cov}(\mathcal{R}) > \omega_1 \), one can easily construct a pseudointersection \( S \) for \( A \) which is a contradiction.

**SPS\(_1\)(\( k \)) =** For every \( A \in [[\omega]^\omega]^{\leq k} \) and for every \( \pi \)-base \( F \subseteq [\omega]^\omega \) for \( A \) there exists an infinite pseudointersection \( S \) of \( A \) meeting each member of \( B \).

**Equal\(_1\)(\( k \)) =** (\( \forall F \in [[\omega]^\omega]^{\leq k} \))(\( \exists h \in \omega \})(\( \forall f \in F \))(\( \exists \infty n \))(\( f(n) = h(n) \)), where \( \exists \infty n \) abbreviates the statement “there are infinitely many”.

**Equal\(_2\)(\( k \)) =** (\( \forall F \in [[\omega]^\omega]^{\leq k} \))(\( \exists h \in \omega \})(\( \forall f \in F \))(\( \exists \infty n \))(\( f(n) = h(n) \)).

**Bounded\(_1\)(\( k \)) =** (\( \forall F \in [[\omega]^\omega]^{\leq k} \))(\( \exists h \in \omega \))(\( \forall f \in F \))(\( \exists \infty n \))(\( f(n) < h(n) \)).

The bounding number \( b \) and the dominating number \( d \) are the least cardinal numbers \( k \) for which the statements Bounded\(_1\)(\( k \)) and Weak Bounded\(_1\)(\( k \)) fail respectively.

Let \( (X, T) \) be a topological space. A dense subset \( Q \) of the poset \( (T \setminus \emptyset, \subseteq) \) is called a \( \pi \)-base for \( X \).

A set \( N \subseteq X \) is called nowhere dense iff \( \text{int}(N) = \emptyset \), and a set \( A \subseteq X \) is called meagre iff \( A \) is the union of countable many nowhere dense sets. The covering number, \( \text{cov}(X) \), and the additivity number, \( \text{add}(X) \), of the space \( X \) are given by:

\[
\text{cov}(X) = \min\{|D| : D \text{ is a family of dense open sets in } X \text{ with } \bigcap D = \emptyset\},
\]

\[
\text{add}(X) = \min\{|D| : D \text{ is a family of meagre sets of } X \text{ but } \bigcup D \text{ is not a meagre set of } X\}.
\]

All undefined terms are used as in [5, 8, 10].

### 2. Introduction and preliminary results

The covering number, \( \text{cov}(\mathcal{R}) \), of the real line may be a singular cardinal number [11]. In this case, it has been shown by A. W. Miller in [13] that \( \text{cov}(\mathcal{R}) \) has uncountable cofinality, and T. Bartoszynski and J. I. Ihoda have shown in [3] that its cofinality is greater than or equal to \( \text{add}(L) \), where \( \text{add}(L) \) is the additivity number of the ideal \( L \) of all Lebesque null sets of the real line. It is an open question (see [3, 4, 15]) whether \( \text{cf}(\text{cov}(\mathcal{R})) \) can be less than \( \text{add}(\mathcal{R}) \).

Note that if \( \text{cov}(\mathcal{R}) \leq b \), then by Lemma 3 we have \( \text{cov}(\mathcal{R}) = \text{add}(\mathcal{R}) \) and that if \( \text{cov}(\mathcal{R}) = d \), then \( \text{cf}(d) \geq b \) (see [5, Theorem 3.1(d), p. 116]). Thus the above-mentioned question is nontrivial only in case \( b \leq \text{cov}(\mathcal{R}) < d \) (it is known (see [7]) that \( \text{cov}(\mathcal{R}) \leq d \)).

The technology that exists in this area seems to be inadequate in answering the above-mentioned question. The aim of this paper is mainly to find new characterisations of the cardinals \( \text{cov}(\mathcal{R}) \) and \( \text{add}(\mathcal{R}) \) and then reformulate the
problem in terms of these new characterisations. Such characterisations may be raised from simple combinatorial statements holding true in the presence of the continuum hypothesis \( \text{CH} \); e.g.,

For every family \( \mathcal{D} \), \( |\mathcal{D}| < \mathfrak{c} \), of dense open sets of the real line \( \mathbb{R} \) there exists a countable dense set \( G \subseteq \mathbb{R} \) such that \( |G \setminus D| < \omega \) for all \( D \in \mathcal{D} \).

For every family \( \mathcal{D} \), \( |\mathcal{D}| < \mathfrak{c} \), of dense open sets of the real line \( \mathbb{R} \) and for every countable dense set \( Q \subseteq \mathbb{R} \) there exists a countable dense set \( G \subseteq Q \) such that \( |G \setminus D| < \omega \) for all \( D \in \mathcal{D} \).

Let us recall some characterisations of \( \text{cov}(R) \) and \( \text{add}(R) \) which we will use in the sequel.

**Lemma 1** ([1], Miller-Bartoszynski). \( \text{cov}(R) > k \) if and only if \( \text{Equal}(k) \).

**Lemma 2** ([6]). \( \text{add}(R) > k \) if and only if for every family \( \mathcal{D} = \{D_i : i \in k\} \) of dense open sets of \( \mathbb{R} \) there exists a family \( \mathcal{Q} = \{Q_n : n \in \omega\} \) of dense open sets of \( \mathbb{R} \) such that for every \( i \in k \) there exists \( n \in \omega \) with \( Q_n \subseteq D_i \).

**Lemma 3** ([14], Miller-Truss). \( \text{add}(R) > k \) if and only if \( \text{Bounded}(k) \) and \( \text{cov}(R) > k \).

As a corollary to Exercise C4 from [10, p. 242], one can easily establish the following lemma.

**Lemma 4.** Let \( (P, \leq) \) and \( (Q, \leq) \) be any two countable nonatomic posets. Then there exist dense sets \( P' \subseteq P \) and \( Q' \subseteq Q \) such that \( (P', \leq) \) and \( (Q', \leq) \) are isomorphic as posets. (There exists a bi-injective function \( H: P' \rightarrow Q' \) such that \( H(p) \leq H(q) \) if and only if \( p \leq q \) for all \( p, q \in P' \).)

### 3. Characterisations of \( \text{cov}(R) \)

**Theorem 1.** The following are equivalent for every cardinal \( k \):

1. \( \text{cov}(R) > k \).
2. \( \text{Equal}(k) \).
3. For every family \( \mathcal{D} \), \( |\mathcal{D}| < k \), of dense open sets of the real line \( \mathbb{R} \) there exists a countable dense set \( G \subseteq \mathbb{R} \) such that \( |G \setminus D| < \omega \) for all \( D \in \mathcal{D} \).
4. For every family \( \mathcal{D} \), \( |\mathcal{D}| < k \), of dense open sets of the real line \( \mathbb{R} \) there exists a countable infinite set \( G \) such that \( |G \setminus D| < \omega \) for all \( D \in \mathcal{D} \).
5. \( \text{Equal}^*(k) \).
6. \( \text{PS}(k) \).
7. \( \text{MA}_k \) (countable) (Martin's Axiom restricted to countable posets).
8. For every family \( \mathcal{D} \), \( |\mathcal{D}| < k \), of dense open sets of the real line \( \mathbb{R} \) and for every countable dense set \( Q \subseteq \mathbb{R} \) there exists a countable infinite set \( G \subseteq Q \) such that \( |G \setminus D| < \omega \) for all \( D \in \mathcal{D} \).
9. For every \( T_1 \) space \( X \) of countable \( \pi \)-weight and every family \( \mathcal{D} = \{D_i : i \in k\} \) of dense open sets in \( X \) there exists a countable infinite set \( G \subseteq X \) such that \( |G \setminus D_i| < \omega \) for all \( i \in k \).

**Proof.** (1) \( \iff \) (2) and (1) \( \iff \) (7) are well known. (1) \( \iff \) (2) is Lemma 1, and (1) \( \iff \) (7) is a consequence of Lemma 4. 

(1) \( \implies \) (3) and (3) \( \implies \) (4) are straightforward.
(4) \to (5). Fix \( F = \{ f_i : i \in k \} \subseteq \omega^\omega \). For every \( i \in k \), \( D_i = \{ f \in \omega^\omega : f_i(n) = f(n) \text{ for some } n \in \omega \} \) is clearly a dense open subset of \( \omega^\omega \). Using (4), Lemma 4, and standard density arguments one can easily verify that there exists a countable infinite set \( G \subseteq \omega^\omega \) such that \( |G \setminus D_i| < \omega \) for all \( i \in k \). It is not hard to see that \( G \) satisfies Equal*\((k)\) for the collection \( F \) as required.

(5) \to (2). First we need to show

**Claim.** Equal*\((k)\) \to Weak Bounded\((k)\).

**Proof.** Let \( F \subseteq \omega^\omega \) be a family of size \( k \). Using the assumption choose a family \( \{ g_n : n \in \omega \} \) such that
\[
(\forall f \in F)(\forall \kappa)(\exists n)(g_n(\kappa) = f(\kappa)).
\]
By passing to a subsequence we can assume that, for every \( n \), \( g_n(n), g_{n+1}(n), g_{n+2}(n), \ldots \) are all equal or pairwise different.

**Subclaim.** For every \( f \in F \) there exists \( n \) and \( \kappa \leq 2n \) such that \( g_\kappa(n) = f(n) \).

**Proof.** Suppose not, and let \( f \in F \) be such that \( g_\kappa(n) \neq f(n) \) for \( \kappa \leq 2n \). Find \( \kappa_0 \) and a sequence \( \{ n_\kappa : \kappa \geq \kappa_0 \} \) such that \( g_\kappa(n_\kappa) = f(n_\kappa) \) for \( \kappa \geq \kappa_0 \).

By the assumption, \( \kappa > 2n_\kappa \). Consider terms \( n_{\kappa_0}, n_{\kappa_0+1}, \ldots, n_{2\kappa_0} \). Note that all these terms are smaller than \( \kappa_0 \). Thus, for some \( i < j \), \( n_{\kappa_0+i} = n_{\kappa_0+j} = n^* \).

It follows that
\[
g_{\kappa_0+i}(n^*) = g_{\kappa_0+j}(n^*) = f(n^*).
\]
In particular, \( g_n(n^*) = f(n^*) \), a contradiction finishing the proof of the subclaim.

To finish the proof of the claim define
\[
g(n) = \max \{ g_\kappa(n) : \kappa \leq 2n \} + 1 \quad \text{for } n \in \omega.
\]
Clearly
\[
(\forall f \in F)(\exists n)(f(n) < g(n)).
\]
This shows that \( F \) cannot be dominating as required.

To complete the proof of (5) \to (2), fix \( F = \{ f_i : i \in k \} \subseteq \omega^\omega \) and let \( A = \{ A_j : j \in \omega \} \subseteq [\omega]^{\omega_\omega} \) be a partition of \( \omega \). For every \( j \in \omega \) let \( G_j = \{ g(j, n) : n \in \omega \} \) satisfy Equal*\((k)\) for the collection \( F_j = \{ f_i | A_j : i \in k \} \). Define a function \( h_j : \omega \to \omega \) by requiring
\[
h_j(j) = \min(\{ m : (\forall l \geq m)(\exists u \in A_j)(g(j, l)(u) = f_i(u)) \}).
\]
Let, by the claim, \( h : \omega \to \omega \) satisfy Weak Bounded\((k)\) for the collection \( H = \{ h_j : i \in k \} \), and define a function \( g : \omega \to \omega \) by letting
\[
g/A_j = g(j, h(j)).
\]
Clearly \( g \) satisfies Equal\((k)\) for the collection \( F \) as required.

(7) \to (6). Fix a set \( A = \{ A_i : i \in k \} \subseteq [\omega]^{\omega} \), and let \( B = \{ b_n : n \in \omega \} \subseteq [\omega]^{\omega} \) be a \( \pi \)-base for \( A \). Clearly \( D_i = \{ b \in B : b \subseteq A_i \} \) is dense in \( (B, \subseteq) \) for all \( i \in k \). Using (7), one can easily verify that there exists a filter \( G = \{ b_n : n \in \omega \} \) of \( (B, \subseteq) \) meeting each \( D_i \) nontrivially. Via an easy induction pick for every \( n \in \omega \),
\[
s_n \in (b_n \setminus \{ s_m : m \in n \}).
\]
Clearly, \( S = \{ s_n : n \in \omega \} \) is an infinite pseudointersection for \( A \) as required.
(6) → (8). Let \( D = \{ D_i : i \in k \} \), \( Q \), and \( B \) be a family of dense open sets, a countable dense set, and a countable base for \( R \), respectively.

Put 
\[ A_i = D_i \cap Q \] 
for all \( i \in k \)

and 
\[ B_b = b \cap Q \] 
for all \( b \in B \).

Clearly \( B^* = \{ B_b : b \in B \} \) is a \( \pi \)-base for \( A = \{ A_i : i \in k \} \). Thus, by (6), there exists an infinite pseudointersection \( G \subseteq Q \) for \( A \). This certainly implies that \( |G \setminus D_i| < \omega \) for all \( i \in k \) as required.

(8) → (4) and (6) → (9) are straightforward.

To finish the proof of the theorem it suffices to show

(9) → (6). Fix \( A = \{ A_i : i \in k \} \subseteq [\omega]^{\omega} \), and let \( B = \{ B_n : n \in \omega \} \subseteq [\omega]^{\omega} \) be a \( \pi \)-base for \( A \). Without loss of generality we may assume that \( \bigcup A = \bigcup B = \omega \). For every \( x, n \in \omega \), we let \( B_n(x) = B_n \setminus \{ x \} \) and put
\[ F = \{ B_n(x) : n, x \in \omega \}. \]

Clearly the topology \( T_F \) which is produced from the subbase \( F \) is \( T_1 \) and second countable. Furthermore, \( A \) is a family of dense open sets in the topological space \( (\omega, T_F) \). Thus by (9) there exists a set \( G \subseteq X \), \( |G| = \omega \), such that \( |G \setminus A_i| < \omega \) for all \( i \in k \) as required. \( \square \)

As an easy corollary of Theorem 1(4) we have \( \text{cf}(\text{cov}(R)) > \omega \). In fact, more than that is true. Namely, let \( s_c \) denote the least \( k \) for which the following statement fails.

For every family \( G \subseteq [\omega]^{\omega} \), \( |G| < \omega \), and every family \( F = \{ f_i : i \in k \} \subseteq [\omega]^{\omega} \) there exists a family \( H = \{ h_n : n \in \omega \} \subseteq [\omega]^{\omega} \) such that for every \( g \in G \) if \( g \) meets all but less than \( k \) many members of \( F \), then \( g \) meets all but finitely many members of \( H \)

Then, we have:

**Corollary 1.** If \( \text{add}(R) \leq s_c \), then \( \text{cf}(\text{cov}(R)) \geq \text{add}(R) \).

**Proof.** Assume on the contrary, and let 
\[ \text{cf}(\text{cov}(R)) = \lambda < \text{add}(R) \leq s_c. \]

In view of [5, Theorem 3.1(d), p. 116], we may assume that \( k = \text{cov}(R) < \omega \).

Fix, by Theorem 1, a family 
\[ G = \{ g_i : i \in k \} \subseteq [\omega]^{\omega} \]

such that 
\[ (\forall H \in [\omega]^{\omega}) (\exists i \in k) (\exists h \in H) (h \cap g_i = \varnothing). \]

Fix \( \{ \lambda_j : j \in \lambda \} \) a cofinal set in \( k \). For every \( j \in \lambda \) let \( f_j : \omega \to \omega \) be such that 
\[ (\forall i \in \lambda_j) (\exists n)(g_i(n) = f_j(n)). \]

Put \( F = \{ f_j : j \in \lambda \} \), and note that 
\[ (\forall i \in k) (\forall j \in \lambda) (f_j \cap g_i \neq \varnothing). \]

Fix, by \( s_c > \lambda \), a set \( H \in [\omega]^{\omega} \) satisfying 
\[ (\forall i \in k) (\exists n)(h \cap g_i \neq \varnothing). \]

This contradicts the choice of \( G \) and finishes the proof. \( \square \)
4. CHARACTERISATIONS OF \( \text{add}(R) \)

**Theorem 2.** The following are equivalent for every cardinal \( k \):

1. \( \text{add}(R) > k \).
2. \( \text{SPS}(k) \).
3. For every family \( D \), \( |D| < k \), of dense open sets of the real line \( R \) and every countable dense set \( Q \subseteq R \) there exists a countable dense set \( G \subseteq Q \) such that \( |G \setminus D| < \omega \) for all \( D \in D \).
4. Equal\(^*(k) \) and Bounded\( (k) \).
5. Equal\( (k) \) and Bounded\( (k) \).
6. MA\( _k \) (countable) and Bounded\( (k) \).
7. For every \( T_1 \) space \( X \) of countable \( \pi \)-weight, every family \( D = \{D_i : i \in k\} \) of dense open sets in \( X \), and every countable dense set \( Q \subseteq X \) there exists a dense set \( G \subseteq Q \) such that \( |G \setminus D_i| < \omega \) for all \( i \in k \).

**Proof.** (1) \( \rightarrow \) (3) follows immediately from Lemma 2.

(3) \( \rightarrow \) (4). In view of Theorem 1 and Lemma 3, we only have to show that (3) implies Bounded\( (k) \). Fix \( F = \{f_i : i \in k\} \subseteq \omega^\omega \). Without loss of generality we may assume that each \( f_i \) is a strictly increasing function. It is easy to see that

\[ D_i = \{f \in \omega^\omega : f(n) > f_i(n) \text{ for some } n \in \omega\} \]

is a dense open set of the Baire space \( \omega^\omega \). Let

\[ G = \{g_n : n \in \omega\} \subseteq \omega^\omega, \]

where \( g_n \) is eventually equal to zero. Clearly \( G \) is dense in \( \omega^\omega \). By (3), Lemma 4, and standard density arguments, there exists a dense set \( Q = \{q_n : n \in \omega\} \subseteq G \) such that

\[ |Q \setminus D_i| < \omega \]

for all \( i \in k \).

Choose a subsequence \( Q' = \{q_{n_v} : v \in \omega\} \) of \( Q \) such that

\[ (\forall u \geq n_v)(q_{n_v}(u) = 0) \land (\forall v \in \omega)((q_{n_{v+1}} | n_v) = 0). \]

On the basis of \( Q' \) we define a function \( f : \omega \rightarrow \omega \) by requiring

\[ f(u) = \max\{q_{n_v}(t) : v \leq u + 1, t \in \text{Dom}(q_{n_v})\}. \]

\( f \) dominates \( F \). Indeed, fix \( i \in k \), and let \( v' \in \omega \) satisfy

\[ (\forall v \geq v')(q_{n_v}(n) > f_i(n)). \]

This means that

\[ (\forall v \geq v')(\exists n \in [n_{v-1}, n_v])(q_{n_v}(n) > f_i(n)). \]

Thus, if \( v > v' \), then we have

\[ f(u) = \max\{q_{n_{v+1}}(n) : n \in [n_u, n_{u+1}]\} > f_i(n_u) > f_i(u), \]

and the desired result follows.

Implications (4) \( \rightarrow \) (5) \( \rightarrow \) (6) \( \rightarrow \) (1) are clear (Theorem 1 and Lemma 3).

(6) \( \rightarrow \) (2). Fix \( A = \{A_i : i \in k\} \subseteq [\omega]^{\omega^\omega} \), and let \( B = \{b_n : n \in \omega\} \subseteq [\omega]^{\omega^\omega} \) be a base for \( A \). For every \( n \in \omega \), by (6), fix \( S_n = \{s(n, m) : n \in \omega\} \subseteq b_n \) an infinite pseudointersection for \( A \). For every \( i \in k \) define a function \( f_i : \omega \rightarrow \omega \) by requiring

\[ f_i(n) = \min\{v : s(n, m) \in D_i \text{ for all } m \geq v\}. \]
Put $F = \{f_i : i \in k\}$. By Lemma 3 ($k < \text{add}(R) \leq b$), there exists a function $f: \omega \to \omega$ such that

$$(\forall i \in k)(\forall^\infty n)(f_i(n) \leq f(n)).$$

On the basis of $f$ we define a set $S$ by letting

$$S = \{s(n, m) : n \in \omega, \ m \geq f(n)\}.$$

It can be readily verified that $S$ is a pseudointersection of $A$ meeting every member of $B$ in an infinite set as required.

(2) $\Rightarrow$ (3). Fix $D = \{D_i : i \in k\}$ and $G$ as in (3). By Lemma 2, there is a family $Q = \{Q_n : n \in \omega\}$ of dense open sets of $R$ such that

$$(\forall i \in k)(\exists n \in \omega)(Q_n \subseteq D_i).$$

Let $B = \{B_n : n \in \omega\}$ be a base for the topology of $R$. By induction, choose a set

$$T = \{t_n : n \in \omega\} \subseteq G$$

such that

$$t_n \in \left(\left(\bigcap\{Q_m : m \leq n\}\right) \cap G\right) \setminus \{t_m : m < n\}.$$

Clearly $T$ is dense in $R$, and $|T \setminus Q_n| < \omega$ for all $n \in \omega$. This, when combined with the above, finishes the proof of (2) $\Rightarrow$ (3).

(7) $\Rightarrow$ (3). It suffices in view of Theorem 1 to show that $\text{PS}(k)$ and $\text{Bounded}(k)$ together imply (2). This can be established as in (6) $\Rightarrow$ (2) of the present theorem. □

Let $\text{PS}(\omega_1, k)$ be the generalisation of $\text{PS}(k)$ to the next higher cardinal with the additional requirement that $B$ be a countable closed base ($\bigcap Q \in B$ for every $Q \in [B]^\omega$) of size $\omega_1$. Of course the continuum hypothesis must be assumed here. Also, let $\omega^* = \beta\omega \setminus \omega$ denote the remainder of the Stone-Čech compactification of $\omega$ with the discrete topology.

**Question 1.** Assume CH. Does $\text{PS}(\omega_1, k)$ imply $\text{cov}(\omega^*) > k$?

**Question 2.** Can $s_e$ be strictly less than $\text{add}(R)$?

**Acknowledgments**

The author wishes to thank Professor D. H. Fremlin for all his communications and the referee for some valuable remarks concerning the notions of $\text{cov}(R)$ and $\text{add}(R)$ and for shaping the paper in its present form.

**References**


Department of Mathematics, University of Aegean, Karlovasi 83200, Samos, Greece

E-mail address: kker@pythagoras.aegean.riadne-t.gr