A LOWER BOUND FOR THE CLASS NUMBERS OF ABELIAN ALGEBRAIC NUMBER FIELDS WITH ODD DEGREE

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Abstract. Let $\Delta_K$, $h_K$, $R_K$ denote the discriminant, the class number, and the regulator of the Abelian algebraic number field $K = \mathbb{Q}(\alpha)$ with degree $d$, respectively. In this note we prove that if $d > 1$, $2 \nmid d$, and the defining polynomial of $\alpha$ has exactly $r_1$ real zeros and $r_2$ pairs of complex zeros, then

$$h_K > \frac{w \sqrt{|\Delta_K|}}{2^n (2\pi)^n 33R_K \log 4|\Delta_K|},$$

where $w$ is the number of roots of unity in $K$.

Let $\Delta_K$, $h_K$, $R_K$ denote the discriminant, the class number, and the regulator of the Abelian algebraic number field $K = \mathbb{Q}(\alpha)$ with degree $d$, respectively. In this note we prove the following result:

Theorem. If $d > 1$, $2 \nmid d$, and the defining polynomial of $\alpha$ has exactly $r_1$ real zeros and $r_2$ pairs of complex zeros, then

$$h_K > \frac{w \sqrt{|\Delta_K|}}{2^n (2\pi)^n 33R_K \log 4|\Delta_K|},$$

where $w$ is the number of roots of unity in $K$.

Upon applying the above theorem, we can improve some known results concerning the lower bound of $h_K$. For instance, Barrucand, Loxton, and Williams [1] proved that if $K = \mathbb{Q}(D^{1/3})$, where $D = n^3 + m$ is not a cube, $m$ and $n$ are nonzero integers with $3n^2 \equiv 0 \pmod{m}$, then

$$h_K > \frac{0.14|\Delta_K|^{1/4}}{\log(|\Delta_K|/3) \log(|\Delta_K|/27)}.$$

Notice that $r_1 = 1$, $r_2 = 1$, $w = 2$, and $R_K < 3 \log(|\Delta_K|/3)$ in this case. By (1), we get a better lower bound as follows:

$$h_K > \frac{\sqrt{|\Delta_K|}}{198\pi \log(|\Delta_K|/3) \log 4|\Delta_K|}.$$

The proof of Theorem. Let $\zeta_K(s)$ denote Dedekind's $\zeta$-function of $K$. By [3, §42], if $\sigma > 1$, then

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

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where \( b_1 = 1 \) and \( b_n \geq 0 \) for \( n > 1 \). Since \( \zeta_K(2) \geq b_1 \) and
\[
(-1)^m r_k^{(m)}(2) = \sum_{n=1}^{\infty} \frac{b_n (\log n)^m}{n^2} \geq 0
\]
for \( m > 0 \), we have
\[
(2) \quad \zeta_K(s) = \sum_{m=0}^{\infty} a_m (2-s)^m, \quad a_0 \geq 1, a_m \geq 0, m > 0, |s - 2| < 1.
\]

Let \( X \) be the group of Dirichlet characters associated to \( K \). It is a well-known fact that \( \zeta_K(s) = \zeta(s) \zeta_K(s) \), where \( \zeta(s) \) is the Riemann \( \zeta \)-function,
\[
\zeta_K(s) = \prod_{\chi \in X} L(s, \chi),
\]
where \( \chi_0 \) is the trivial character and \( L(s, \chi) \) is the \( L \)-series attached to the character \( \chi \). Since \( \zeta_K(s) \) has only simple pole at \( s = 1 \) of residue \( \zeta_K(1) \), the function \( g(s) = \zeta_K(s) - \zeta_K(1)/(s - 1) \) is regular. From (2), we get
\[
(3) \quad g(s) = \sum_{m=0}^{\infty} (a_m - \zeta_K(1))(2-s)^m.
\]

For any \( \sigma > 0 \) and any \( x \geq 1 \), using Abel’s transformation,
\[
L(s, \chi) = \sum_{1 \leq n \leq x} \frac{\chi(n)}{n^s} - \frac{S(x, \chi)}{x^s} + s \int_{x}^{\infty} \frac{S(z, \chi)}{z^{s+1}} \, dz,
\]
where \( S(x, \chi) = \sum_{1 \leq n \leq x} \chi(n) \). Let \( f_\chi \) denote the conductor of \( \chi \). By Pólya’s theorem, \( |S(x, \chi)| < \sqrt[n]{f_\chi} \log f_\chi \). By (4), we get
\[
(5) \quad |L(s, \chi)| \leq \sum_{1 \leq n \leq x} \frac{1}{n^\sigma} + \frac{\sqrt[n]{f_\chi} \log f_\chi}{x^\sigma} < 1 + \frac{x^{1-\sigma} - 1}{1-\sigma} + \frac{2\sqrt[n]{f_\chi} \log f_\chi}{x^\sigma}.
\]
Putting \( x = \sqrt[n]{f_\chi} \log f_\chi \). We get from (5) that
\[
(6) \quad |L(s, \chi)| < 4 f_\chi^{1/4} \sqrt[n]{\log f_\chi} < f_\chi^{5/4}, \quad \sigma \geq \frac{1}{2},
\]
since \( f_\chi \geq 5 \). Furthermore, by the conductor-discriminant formula
\[
(7) \quad \Delta_K = (-1)^r \prod_{\chi \in X} f_\chi,
\]
we get from (6) that
\[
(8) \quad |\zeta_K(s)| < \prod_{\chi \in X, \chi \neq \chi_0} f_\chi^{5/4} = |\Delta_K|^{5/4}, \quad \sigma \geq \frac{1}{2}.
\]
Simultaneously, since \( |\zeta(s)| \leq 1/|s - 1| + |s|/|s| \) for \( \sigma > 0 \), we have
\[
|\zeta(s)| \leq 3, \quad |s - 2| = \frac{3}{2}.
\]
Therefore, by (8),

\[ |\zeta_K(s)| < 3|\Delta_K|^{5/4}, \quad |s - 2| = \frac{3}{2}. \]

Further, by a well-known fact that \(|L(1, \chi)| < \log f_X + 2\), we get from (9) that

\[ |g(s)| \leq |\zeta_K(s)| + \left| \frac{\xi_K(1)}{s - 1} \right| < 4|\Delta_K|^{5/4} \]

for \(|s - 2| = \frac{3}{2}\). Furthermore, by the maximum modulus principle, (10) holds for \(|s - 2| \leq \frac{3}{2}\). Using Cauchy's theorem, we find from (3) and (10) that

\[ |a_m - \xi_K(1)| < 4|\Delta_K|^{5/4} \left( \frac{2}{3} \right)^m, \quad m \geq 0. \]

Let \(M\) be an integer with \(M > 1\). By (2) and (11), if \(13/14 < \sigma < 1\), then

\[ g(\sigma) = \zeta_K(\sigma) - \xi_K(1) \frac{1}{\sigma - 1} \]

\[ \leq \sum_{m=0}^{M-1} (a_m - \xi_K(1))(2 - \sigma)^m - \sum_{m=M}^{\infty} |a_m - \xi_K(1)|(2 - \sigma)^m \]

\[ < 1 - \xi_K(1) \sum_{m=0}^{M-1} (2 - \sigma)^m - 4|\Delta_K|^{5/4} \sum_{m=M}^{\infty} \left( \frac{2}{3} (2 - \sigma) \right)^m \]

\[ < 1 - \xi_K(1) \frac{(2 - \sigma)^M - 1}{1 - \sigma} - 14|\Delta_K|^{5/4} \left( \frac{2}{3} \right)^M. \]

Put

\[ M = \left\lceil \frac{\log(140|\Delta_K|^{5/4})}{\log(7/5)} \right\rceil + 1. \]

We get from (12) and (13) that

\[ \zeta_K(\sigma) > \frac{9}{10} - \frac{(2 - \sigma)^M}{1 - \sigma} \xi_K(1). \]

By a recent result of Chen and Wang [2], if \(\chi\) is a complex character, then \(L(s, \chi)\) has no zero in the range

\[ 1 - 1/9.0321 \log_4|\Delta_K| < \sigma < 1. \]

Since \(c > 0.0553581\) for \(f_X \geq 5\), we see that \(\triangle(\sigma, \chi) \neq 0\) for the range \(1 - 1/18.0642 \log_2 f_X \leq \sigma < 1\). Since \(2 \nmid d\), all characters of \(\chi\) are complex characters. Notice that \(d \geq 3\). We get from (7) that \(|\Delta_K| \geq f_X^2\) for any \(\chi \in \chi\). Hence, \(\triangle(\sigma, \chi) \neq 0\) for \(1 - 1/9.0321 \log_4|\Delta_K| \leq \sigma < 1\). It implies that \(\xi_K(\sigma) < 0\) for \(1 - 1/9.0321 \log_4|\Delta_K| \leq \sigma < 1\), and by (14), we obtain

\[ \zeta_K(\sigma) > \frac{9(1 - \sigma_0)}{10(2 - \sigma_0)^M}, \]

where

\[ c = \max \left( \frac{0.089193}{0.339} \right), \quad t > 0. \]

Since \(c > 0.0553581\) for \(f_X \geq 5\), we see that \(\triangle(\sigma, \chi) \neq 0\) for the range \(1 - 1/18.0642 \log_2 f_X \leq \sigma < 1\). Since \(2 \nmid d\), all characters of \(\chi\) are complex characters. Notice that \(d \geq 3\). We get from (7) that \(|\Delta_K| \geq f_X^2\) for any \(\chi \in \chi\). Hence, \(\triangle(\sigma, \chi) \neq 0\) for \(1 - 1/9.0321 \log_4|\Delta_K| \leq \sigma < 1\). It implies that \(\xi_K(\sigma) < 0\) for \(1 - 1/9.0321 \log_4|\Delta_K| \leq \sigma < 1\), and by (14), we obtain

\[ \zeta_K(\sigma) > \frac{9(1 - \sigma_0)}{10(2 - \sigma_0)^M}, \]
where \( \sigma_0 = 1 - 1/9.0321 \log 4|\Delta_K| \). Since \( d \geq 3 \) and \( |\Delta_K| \geq 23 \), we get from (13) that

\[
\log(2 - \sigma_0)^M = M \log \left(1 + \frac{1}{9.0321 \log 4|\Delta_K|}\right) < \left(1 + \frac{\log 140|\Delta_K|^{5/4}}{\log(7/5)}\right) \left(\frac{1}{9.0321 \log 4|\Delta_K|}\right) \\
\leq \left(1 + \frac{\log 140 + \log 23^{5/4}}{\log(7/5)}\right) \left(\frac{1}{9.0321 \log 4.23}\right) < 1.17
\]

and \( (2 - \sigma_0)^M < 3.23 \). Substituting it into (15),

\[
(16) \quad \xi_K(1) > \frac{1}{33 \log 4|\Delta_K|}.
\]

Thus, by (16) and the class number formula

\[
h_K = \frac{w \sqrt{|\Delta_K|}}{2^{n/2} (2\pi)^{n/2} R_K} \xi_K(1),
\]

we get (1) immediately. The theorem is proved.

\section*{References}


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