TWISTING OPERATIONS AND COMPOSITE KNOTS

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Abstract. Suppose that a composite knot $K$ in $S^3$ can be changed to a trivial knot by $1/n$-surgery along a trivial loop $C$. We show that $|n| \leq 2$. Moreover, if there is a decomposing sphere of $K$ which meets $C$ in two points, then $|n| \leq 1$.

1. Introduction

Let $K$ be a knot in the 3-sphere $S^3$ and $D$ a disk which intersects $K$ transversely in its interior. Let $C = \partial D$. We get a new knot $K^*$ in $S^3$ as the image of $K$ after doing $1/n$-surgery along $C$. We say that $K^*$ is obtained from $K$ by $n$-twisting along $C$. In particular, this operation is called a trivializing $n$-twist of $K$ if $K^*$ is unknotted. We remark that a crossing change is equivalent to $\pm 1$-twist on a disk which intersects $K$ in precisely two points.

In [4], Mathieu asked if there is a composite knot which admits a trivializing twist. Several families of composite knots are known to admit trivializing twists at present [5], [7], [11]. Since all the examples of trivializing twists of composite knots are $\pm 1$-twists, it is conjectured that if a composite knot admits a trivializing $n$-twist, then $|n| \leq 1$ [6]. In fact, Motegi [6] proved that $|n| \leq 5$, by making use of Gordon’s result about Dehn fillings on hyperbolic manifolds [2].

In this paper we improve Motegi’s result as follows.

Theorem 1. If a composite knot $K$ admits a trivializing $n$-twist, then $|n| \leq 2$.

The possibility of $|n| = 2$ remains an open problem.

If a knot $K$ is composite, then there is a 2-sphere $S$ which intersects $K$ transversely in two points, such that each one of the 3-balls bounded by $S$ intersects $K$ in a knotted spanning arc. Such a sphere is called a decomposing sphere of $K$.

Theorem 2. Suppose that a composite knot $K$ admits a trivializing $n$-twist along $C$ and that there is a decomposing sphere $S$ of $K$ which intersects $C$ transversely in two points. Then $|n| \leq 1$.

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It is easy to verify that all known examples in the above papers satisfy this assumption. An example is illustrated in Figure 1.

Scharlemann [9] proved that unknotting number one knots are prime. That is, a composite knot cannot be trivialized by ±1-twists on a disk which meets the knot in two points. (See also [10].) Miyazaki-Yasuhara [5] found many examples of composite knots which do not admit trivializing twists.

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2. Preliminaries

Let $K$ be a composite knot in $S^3$. Suppose that $K$ admits a trivializing $n$-twist along $C$. Let $M = S^3 - \text{Int } N(K \cup C)$. Let us write $T = \partial N(C), T' = \partial N(K)$. Slopes on $T$ or $T'$ will be parametrized by $\mathcal{S} \cup \{1/0\}$ in the usual way (cf. [8]), using a meridian-longitude basis. Since $K \cup C$ is unsplittable in $S^3$, $M$ is irreducible. For any slope $r$ on $T$, let $M(r)$ denote the manifold obtained from $M$ by $r$-Dehn filling on $T$, that is, by attaching a solid torus $J$ to $M$ along $T$ so that $r$ bounds a disk in $J$. It is immediate from the definitions that $K$ is trivialized by $n$-twisting along $C$ if and only if $M(1/n)$ is a solid torus. Note that $M(1/0) = S^3 - \text{Int } N(K)$.

Let $S$ be a decomposing sphere of $K$. Isotope $S$ so that $S \cap N(C)$ is a disjoint union of meridian disks of $N(C)$ and $m = |S \cap N(C)|$ is minimal. Note that $m \geq 2$. Then $P = S \cap M$ is an incompressible planar surface in $M$, with two outer boundary components $\partial_0 P, \partial_\infty P$, lying in $T'$, and $m$ inner boundary components $\partial_i P$, $i = 1, \ldots, m$, lying in $T$. Here, the inner
boundary components are numbered so that they are consecutive on $T$. Each component of $\partial P$ has slope $1/0$ in $T$ or $T'$.

Let $D_0$ be a meridian disk of $M(1/n)$. Isotope $D_0$ so that $D_0 \cap J$ is a disjoint union of meridian disks of $J$. We choose $D_0$ so that $l = |D_0 \cap J|$ is minimal over all meridian disks of $M(1/n)$. Note that $l \geq 2$. If $l = 1$, then we regard $J$ as a regular neighborhood of a core of $M(1/n)$. This would imply that $M(1/0)$ is a solid torus. From the minimality of $l$, $Q = D_0 \cap M$ is an incompressible planar surface in $M$, with one outer boundary component $\partial_0 Q$, lying in $T'$, and $l$ inner boundary components $\partial_j Q$, $j = 1, \ldots, l$, each having slope $1/n$ in $T$. The inner boundary components are numbered consecutively on $T$. It is easy to see that $\partial_0 Q$ has slope $n\omega^2/1$, where $\omega = \text{lk}(K, C)$.

By an isotopy of $Q$, we may assume that $P$ and $Q$ intersect transversely, and each outer boundary component of $P$ intersects $\partial_0 Q$ exactly once, and each inner boundary component of $P$ intersects each inner boundary of $Q$ in $|n|$ points. Thus, for example, when we go around an inner boundary component of $P$, we will consecutively meet $\partial_1 Q, \partial_2 Q, \ldots, \partial_l Q, \ldots, \partial_1 Q, \ldots, \partial_l Q$ (repeated $|n|$ times). By an innermost argument, we can assume that no loop component of $P \cap Q$ bounds a disk in $P$ or $Q$, since $P$ and $Q$ are incompressible and $M$ is irreducible.

As in [1], we form the associated graphs $G_P$ and $G_Q$. Let $A$ be the annulus obtained by capping off the inner boundary components of $P$ by meridian disks of $N(C)$. We obtain a graph $G_P$ in $A$ by taking as the “fat” vertices of $G_P$ the disks in $N(C)$ that cap off the inner boundary components of $P$, and as the edges of $G_P$ the arc components of $P \cap Q$ in $P$. Similarly we obtain the graph $G_Q$ in the disk $D_0$.

Let $G$ denote either $G_P$ or $G_Q$.

If an edge $e$ connects a vertex to a vertex, then $e$ is an interior edge; otherwise, it is a boundary edge. Note that $G$ has at most two boundary edges. If $G_P$ has two boundary edges, so does $G_Q$, and vice versa. Each vertex of $G_P$ ($G_Q$) has degree $|n|/ (|n| m$, resp.).

Let $e$ be an edge of $G_P$. If an end point of $e$ is in $\partial P \cap \partial_j Q$, then we give this end point of $e$ the label $j$. Thus each incidence of an edge of $G_P$ at a vertex of $G_P$ is labeled with a vertex of $G_Q$. Similarly in $G_Q$, label the end points of edges incident to vertices.

Two vertices of $G_P$ ($G_Q$) are parallel if the corresponding inner boundary components of $P(Q)$, when given the orientations induced by some orientation of $P(Q)$, are homologous in $T$; otherwise, they are antiparallel. Since $M$ is orientable, we have the parity rule:

An interior edge $e$ of $G_P$ connects parallel vertices in $G_P$ if and only if $e$ connects antiparallel vertices in $G_Q$.

An $x$-cycle in $G$ is a cycle $\sigma$ of edges in $G$ such that all the vertices of $G$ in $\sigma$ are parallel and $\sigma$ can be oriented so that the tail of each edge has label $x$. A Scharlemann cycle in $G$ is an $x$-cycle $\sigma$ in $G$ for some label $x$ such that $\sigma$ bounds a disk face of $G$. In particular, a Scharlemann cycle of length 1 will be called a trivial loop.

**Lemma 1.** $G$ contains no trivial loops.

**Proof.** This follows immediately from the minimality of $l$ or $m$.

**Lemma 2.** $G$ contains no Scharlemann cycles.
The proof is analogous to [1, proof of Lemma 2.5.2] or [3, proof of Lemma 3.3]. We omit the details.

3. PROOFS

To find Scharlemann cycles, we consider the following conditions as in [1]:

(*) There exists a vertex $x$ of $G$ such that for each label $y$ there is an edge of $G$ incident to $x$ with label $y$, connecting $x$ to an antiparallel vertex of $G$.

(**) For each vertex $x$ of $G$ there exists a label $y(x)$ such that each edge of $G$ incident to $x$ with label $y(x)$ connects $x$ either to a parallel vertex of $G$ or to an outer boundary.

In fact, (**) is the negation of (*).

**Lemma 3.** Suppose that $G_P$ satisfies (*). Then $G_Q$ contains a Scharlemann cycle.

**Proof.** See [1, Lemmas 2.6.2 and 2.6.3].

**Remark.** In general, we cannot exchange the roles of $P$ and $Q$ in the statement of Lemma 3. Because an $x$-cycle in $G_P$ does not necessarily bound a disk in the annulus $A$. However, when $G_P$ has only one boundary edge, we can conclude that $G_P$ contains a Scharlemann cycle if $G_Q$ satisfies (*).

**Lemma 4.** Let $x$ be a vertex of $G_P$. If there exist successive $l$ edges of $G_P$ connecting $x$ to antiparallel vertices, then $G_Q$ contains a Scharlemann cycle.

**Proof.** This follows immediately from Lemma 3.

**Lemma 5.** If $G$ contains a parallel family of edges connecting parallel vertices, then either the sets of labels at the two ends of the family are disjoint, or $G$ contains a Scharlemann cycle. In particular, if $G_P$ ($G_Q$) contains a parallel family of more than $l/2$ ($m/2$, resp.) edges connecting parallel vertices, then $G_P$ ($G_Q$) contains a Scharlemann cycle.

**Proof.** See [1, Lemma 2.6.6 and Corollary 2.6.7].

**Lemma 6.** Suppose that $|n| \geq 2$ and that $G_Q$ satisfies (**). Then either $G_Q$ contains a Scharlemann cycle or every vertex of $G_P$ belongs to a boundary edge of $G_P$.

**Proof.** This is essentially [1, Lemma 2.6.4]. The proof works well even if $G_P$ is a graph in an annulus.

We remark that the latter conclusion of Lemma 6 implies that $G_P$ has exactly two vertices.

Now suppose that $G_P$ satisfies (**). Let $v$ be a vertex of $G_P$. There exists a label $y(v)$ such that each one of $|n|$ edges of $G_P$ incident to $v$ with label $y(v)$ connects $v$ either to a parallel vertex or to $\partial A$. Fix the label $y(v)$. These $|n|$ edges will be called the $y(v)$-edges at $v$. A corner at $v$ is an interval on the boundary of the fat vertex $v$ between successive labels $y(v)$. There are $|n|$ corners around $v$, and there are $l-1$ incidences of edges to $v$ in the interior of a corner. Let $\Gamma = G_P - \{\text{boundary edges}\}$. Let $\overline{\Gamma}$ be the reduced graph of $\Gamma$, obtained by amalgamating all mutually parallel edges in the obvious way. Then $G_P$, $\Gamma$, and $\overline{\Gamma}$ have the same vertex set.

We now want to estimate the degree $\deg_{\overline{\Gamma}}(v)$ of $v$ in $\overline{\Gamma}$.
Lemma 7. Suppose that $G_P$ satisfies (**). Let $v$ be a vertex of $G_P$, and let $b(v)$ be the number of boundary edges incident to $v$. Then $\deg_{\overline{T}}(v) \geq 2|\pi| - b(v)$.

Proof. By Lemmas 2 and 5, any pair of $y(v)$-edges is not parallel. Hence the $y(v)$-edges, except for boundary edges, correspond to distinct edges of $\overline{T}$. Also, not all the $l - 1$ edges incident to $v$ in the interior of a corner are parallel to $y(v)$-edges. Therefore, the interior of a corner yields at least one edge of $\overline{T}$ unless it does not meet a boundary edge. The conclusion follows from these observations.

In fact, we have three possibilities, according to the situation in $G_P$:

1) No boundary edge is incident to $v$.
2) Only one boundary edge is incident to $v$.
3) Two boundary edges are incident to $v$.

There is at most one vertex of $G_P$ that satisfies (3). If a vertex satisfies (2), then $G_P$ has precisely two such vertices.

The following lemma is an easy consequence of Lemma 7 and the observation above.

Lemma 8. Let $\Lambda$ be a component of $\overline{T}$. Let $V$ and $E$ be the number of vertices and edges of $\Lambda$. Then $|\pi| V \leq E + 1$.

Possibly, $G_P$ is disconnected. Choose a point $z \in \partial A - G_P$. We define a partial ordering on the set of components of $G_P$ as in [1]. For two components $H_1$ and $H_2$ of $G_P$, $H_1 < H_2$ if and only if every path in $A$ from $H_1$ to $z$ meets $H_2$. A component of $G_P$ is extremal if it is minimal with respect to the partial ordering for some choice of $z$.

Proof of Theorem 1. Suppose that $|\pi| \geq 3$. If $G_P$ satisfies (*), then $G_Q$ would contain a Scharlemann cycle by Lemma 3, contradicting Lemma 2. Thus $G_P$ satisfies (**).

We may assume that $G_P$ is connected. If $G_P$ is disconnected, we will replace $G_P$ by an extremal component. (We avoid a component without vertex.) We consider the reduced graph $\overline{\Gamma}$ of $\Gamma = G_P - \{\text{boundary edge}\}$ as before. Since $G_P$ is connected, $\overline{\Gamma}$ is also connected. Let $V$, $E$, and $F$ be the number of vertices, edges, and faces of $\overline{\Gamma}$. We do not count the region meeting a component of $\partial A$ as a face of $\overline{\Gamma}$. By Lemma 8, $3V \leq E + 1$. Since $\overline{\Gamma}$ has no 1-sided faces or parallel edges, every face has at least three sides. Let $F_0$ and $F_\infty$ be the components of $A - \overline{\Gamma}$ containing $\partial_0 P$ and $\partial_\infty P$ respectively (possibly, $F_0 = F_\infty$). The frontiers $Fr_{F_0}$ and $Fr_{F_\infty}$ can be expressed as the unions of a sequence of edges. Let $a$ and $b$ be the number of edges in $Fr_{F_0}$ and $Fr_{F_\infty}$ respectively. Note that a double edge is counted twice. Then $3F + a \leq 2E$ if $F_0 = F_\infty$, or $3F + (a + b) \leq 2E$ if $F_0 \neq F_\infty$. By Euler's formula, $1 = V - E + F \leq \frac{1-a}{3}$ if $F_0 = F_\infty$, or $0 = V - E + F \leq \frac{1-(a+b)}{3}$ if $F_0 \neq F_\infty$. This is a contradiction in either case. This completes the proof.

Proof of Theorem 2. Suppose that $|\pi| \geq 2$. Since $|S \cap N(C)| = 2$, $G_P$ has exactly two vertices $x$ and $y$ that are antiparallel. If $G_P$ has only one boundary edge, then the arc corresponding to the boundary edge is essential in the annulus $A$. If $G_Q$ satisfies (*), then $G_P$ contains a Scharlemann cycle by the remark after Lemma 3. If $G_Q$ satisfies (**), then Lemma 6 implies that $G_Q$ contains a Scharlemann cycle, since no vertex of $G_P$ belongs to a boundary edge. In either
case, Lemma 2 gives a contradiction. Hence $G_P$ has exactly two boundary edges.

If two boundary edges are incident to the same vertex $x$, say, then there is a loop $\sigma$ at $y$, since $x$ and $y$ have the same degree. However, $\sigma$ bounds a disk which does not contain the vertex $x$. Hence, there would be a trivial loop. This contradicts Lemma 1. Thus each vertex belongs to a single boundary edge.

We distinguish two cases.

(1) $G_P$ contains no loops.

Then all the interior edges incident to $x$ connect vertices $x$ and $y$. By Lemma 4, $G_Q$ contains a Scharlemann cycle. This contradicts Lemma 2.

(2) $G_P$ contains a loop.

There is a loop based at $x$. Any loop must be essential in $A$. Consider the edge $e$ incident to $x$ immediately to the right of the boundary edge. Then $e$ must be a loop. Otherwise, a loop based at $x$ would be inessential in $A$. Then, without loss of generality, we have a situation as in Figure 2.

Suppose that there are $s$ parallel loops, including $e$. Then by Lemma 5, $s \leq l/2$. But if $s = l/2$, then a loop has the same label at both ends, which contradicts the parity rule. Therefore, $2s + 1 \leq l$. Hence, there are at least $l$ edges connecting $x$ to $y$, since $x$ has degree $|n|/ \geq 2l$. Then, by Lemma 4, $G_Q$ contains a Scharlemann cycle, a contradiction. This completes the proof.

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