THE GAUSSIAN-WAHL MAP FOR TRIGONAL CURVES

JAMES N. BRAWNER

(Communicated by Eric Friedlander)

Abstract. If a curve $C$ is embedded in projective space by a very ample line bundle $L$, the Gaussian map $\Phi_{C,L}$ is defined as the pull-back of hyperplane sections of the classical Gauss map composed with the Plücker embedding. When $L = K$, the canonical divisor of the curve $C$, the map is known as the Gaussian-Wahl map for $C$. We determine the corank of the Gaussian-Wahl map to be $g + 5$ for all trigonal curves (i.e., curves which admit a 3-to-1 mapping onto the projective line) by examining the way in which a trigonal curve is naturally embedded in a rational normal scroll.

1. Introduction

(1.1) If $L$ is a line bundle on a projective variety $X$, we can define the Gaussian map

$$\Phi_L : \Lambda^2 H^0(X, L) \to H^0(X, \Omega^1_X \otimes L^2)$$

essentially by $\Phi_L(f \wedge g) = f \, dg - g \, df$. The name Gaussian derives from the case where $X$ is a curve and $L$ is very ample; in this case $\Phi_L$ is simply the pull-back of hyperplane sections of the classical Gauss map, after composing with the Plücker embedding. When $X$ is a curve and $L = K_X$, the canonical divisor of the curve, Wahl proved the following remarkable theorem concerning the map $\Phi_K$, which we refer to as the Gaussian-Wahl map.

Theorem 1.2 (Wahl [Wa]). If a non-singular curve lies on a K3 surface, then its Gaussian-Wahl map $\Phi_K$ is not surjective.

Other authors have shown that the Gaussian-Wahl map is surjective for the general curve of sufficiently high genus [CHM], and have studied the corank of the map for certain families of curves for which the map is not surjective. Ciliberto and Miranda have shown that $\text{corank}(\Phi_K) = g + 5$ for the general trigonal curve of genus $g \geq 4$ [CM]. They went on to exhibit a specific class of cyclic trigonal curves, defined by the equation $y^3 = x^3L - 1$, for which the corank of $\Phi_K$ is equal to this same generic value. We prove in this paper that in fact this is the case for every trigonal curve of genus $g \geq 4$. After some preliminary discussion we prove the main theorem in §3.
Theorem 1.3. The corank of the Gaussian-Wahl map is \( g + 5 \) for every nonsingular trigonal curve of genus \( g \geq 4 \).

2. Trigonal curves

(2.1) In this section we recall several well-known facts about trigonal curves and rational normal scrolls which we will use in the proof of the main theorem (cf. [Co, Ma]). A non-singular trigonal curve \( C \) of genus \( g \), canonically embedded in \( \mathbb{P}^{g-1} \), lies on a rational normal scroll whose invariants are determined by the \( g_1^3 \). Specifically, if \( k \) is the unique positive integer such that \( h^0((k+1)g_1^1) = k + 2 \) and \( h^0((k+2)g_1^1) \geq k + 4 \), then \( C \) lies on the scroll \( S_{k,l} \subset \mathbb{P}^{k+l+1} \), where \( k + l = g - 2 \) and \( k \leq l \). Maroni showed that this invariant \( k \) is bounded by

\[
\frac{g - 1}{d} \leq k + 1 \leq \frac{g}{2}
\]

and that the ruling of the scroll cuts out the \( g_1^1 \) on the curve [Ma]. Where convenient we will denote the scroll simply as \( S \).

(2.2) The scroll \( S_{k,l} \), which is isomorphic to the surface \( F_n, \ n = l - k \), has Picard group generated by a hyperplane section \( H \) and a fibre of the ruling \( R \), with the following intersections: \( H^2 = k + l \), \( H \cdot R = 1 \), and \( R^2 = 0 \). If \( k < l \), there is a unique curve \( B \) of negative self-intersection on the scroll with class \( B \equiv H - lR \) and intersections \( B^2 = -(l-k) \) and \( B \cdot R = 1 \). The class of the curve \( C \) and the canonical divisor of the scroll \( K_S \) are easily computed, as are those of two other divisors that we will use later:

\[
\begin{align*}
C &\equiv 3H + (4-g)R, \\
K_S &\equiv -2H + (g-4)R \equiv -2B - (n+2)R, \\
K_S + C &\equiv H \equiv B + lR, \\
2K_S + C &\equiv -H + (g-4)R \equiv -B + (k-2)R.
\end{align*}
\]

(2.3) We collect the following computations about global sections of sheaves over the scroll \( S \) (cf. [DM 3.2, 3.3]). If \( D \equiv rB + sR \) is a divisor on a rational normal scroll \( S_{k,l} \equiv F_{l-k} = F_n \), then

(i) \( H^0(S, \mathcal{O}_S(D)) = 0 = H^0(S, \Omega^1_S(D)) \) if \( r < 0 \) or \( s < 0 \).

(ii) \( h^0(S, \Omega^1_S(D)) = 2rs - nr^2 - 2 \) if \( r \geq 1 \) and \( s \geq nr + 1 \).

3. The proof of the theorem

(3.1) If the trigonal curve \( C \) has genus \( g = 4 \), then it is a complete intersection curve, which we handle with a separate argument at the end of the proof; for the remainder, we assume \( g \geq 5 \). From the inclusion \( C \subset S \), we have the following diagram involving the Gaussian surface map \( \Phi_{S,K+C} \):

\[
\begin{array}{ccc}
\Lambda^2 H^0(S, K_S + C) & \xrightarrow{\Phi_{S,K+C}} & H^0(S, \Omega^1_S(2K_S + 2C)) \\
\text{Res} & & \gamma \\
\Lambda^2 H^0(C, \Omega^1_C) & \xrightarrow{\Phi_K} & H^0(C, \Omega^1_C \otimes^3)
\end{array}
\]
First notice that the restriction map \( H^0(S, K_S + C) \to H^0(C, \Omega^1_C) \) is surjective because the scroll \( S \) is a regular surface; consequently the left vertical map \( \text{Res} \) is surjective as well. We proceed to prove the theorem by showing that the top map is surjective and the right vertical map \( \gamma : H^0(S, \Omega^1_S(2K_S + 2C)) \to H^0(C, \Omega^{1 \otimes 3}_C) \) is injective. The rank of the Gaussian-Wahl map \( \Phi_K \) is then equal to the dimension of the space \( H^0(S, \Omega^1_S(2K_S + 2C)) \), which we easily compute to be \( 4g - 10 \), and the theorem follows.

**Proposition 3.2.** \( \Phi_{S,K+C} \) is surjective with \( \text{rank} = 4g - 10 \).

**Proof.** Duflot and Miranda have calculated when the Gaussian map \( \Phi_{S,D} \) is surjective for a divisor \( D \) on a rational normal scroll \( S_{k,l} \cong F_{l-k} = F_n \) by explicitly writing bases for \( H^0(S, \mathcal{O}_S(D)) \) and \( H^0(S, \Omega^1_S(2D)) \). The first part of the proposition follows immediately from the following result [DM, 4.5].

**Theorem 3.3 (Duflot-Miranda).** The Gaussian surface map

\[
\Phi_{S,D} : H^0(S, \mathcal{O}_S(D)) \to H^0(S, \Omega^1_S(2D))
\]
is surjective for a divisor \( D \equiv rB + sR \) on \( S_{k,l} \cong F_n \) if \( r \geq 0 \) and \( s \geq nr + 1 \).

In our case, \( D \equiv K_S + C \equiv B + lR \), so \( r = 1 \geq 0 \) and \( s = l \). If \( g \geq 5 \), then Maroni’s inequality shows that \( k \geq 1 \); consequently \( nr + 1 = l - k + 1 \leq l = s \) and the theorem applies. Therefore the rank of \( \Phi_{S,K+C} \) is simply the dimension of the target space, which we compute using (2.3):

\[
\text{rank}(\Phi_{S,K+C}) = h^0(S, \Omega^1_S(2K_S + 2C)) = h^0(S, \Omega^1_S(2B + 2lR)) = 2(2)(2l) - (l - k)4 - 2 = 4(l + k) - 2 = 4g - 10
\]
since \( r = 2 \geq 1 \) and \( s = 2l \geq 2(l - k) + 2 \). □

**Proposition 3.4.** \( \gamma : H^0(S, \Omega^1_S(2K_S + 2C)) \to H^0(C, \Omega^{1 \otimes 3}_C) \) is injective.

**Proof.** Consider the following two exact sequences associated with the log complex of \( C \subset S \):

\[
0 \to \Omega^1_S \to \Omega^1_S(\log C) \to \mathcal{O}_C \to 0,
0 \to \Omega^1_S(\log C)(-C) \to \Omega^1_S \to \Omega^1_C \to 0.
\]

We twist and take global sections to construct the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
H^0(S, \Omega^1_S(2K_S + C)) & \longrightarrow & H^0(S, \Omega^1_S(\log C)(2K_S + C)) & \longrightarrow & H^0(C, \mathcal{O}_C(2K_S + C)) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(S, \Omega^1_S(2K_S + C)) & \longrightarrow & H^0(S, \Omega^1_S(2K_S + 2C)) & \longrightarrow & H^0(C, \Omega^{1 \otimes 3}_C(2K_S + 2C) \mid C) \\
\gamma \downarrow & & \alpha \downarrow & & \beta \downarrow \\
H^0(C, \Omega^{1 \otimes 3}_C) & = & H^0(C, \Omega^{1 \otimes 3}_C)
\end{array}
\]
The right vertical sequence is obtained from a twist of the standard conormal sequence
\[ 0 \to \mathcal{O}_C(-C) -\mathcal{O}^1_{S|C} \to \mathcal{O}^1_{C} -\to 0. \]
We show that \( \ker(y) = H^0(S, \mathcal{O}^1_{S}(\log C)(2K_S + C)) = 0 \).

(3.5) Let \( \pi : S \to \mathbb{P}^1 \) be the map associated with the \( \mathbb{P}^1 \)-bundle \( S \), \( B \), and \( R \) generators of \( \text{Pic}(S) \) as above, and let \( Z \) be the 0-dimensional subscheme of \( S \) of ramification points of the restriction map \( \pi|_C : C \to \mathbb{P}^1 \). Since \( K_S = -2B-(n+2)R \) and \( \Omega^1_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \), we have the following exact sequence:
\[ 0 \to \pi^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_{S} \to \mathcal{O}_S(-2B - nR) \to 0. \]
Furthermore, since \( \Omega^1_{S}(\log C)(-C) \subset \Omega^1_{S} \), we can construct the following diagram involving the ideal sheaf of \( Z \):
\[
\begin{array}{cccccc}
0 & \to & \pi^*\Omega^1_{\mathbb{P}^1}(-C) & \to & \Omega^1_{S}(\log C)(-C) & \to & \mathcal{I}_Z \cdot \mathcal{O}_S(-2B - nR) & \to & 0 \\
& & \cup & & \cup & & \\
0 & \to & \pi^*\Omega^1_{\mathbb{P}^1} & \to & \Omega^1_{S} & \to & \mathcal{O}_S(-2B - nR) & \to & 0 \\
\end{array}
\]
Twisting the top sequence by \( 2K_S + 2C = 2B + 2lR \) and taking global sections gives
\[
0 \to H^0(S, \pi^*\Omega^1_{\mathbb{P}^1}(2K_S + C)) \to H^0(S, \mathcal{O}^1_{S}(\log C)(2K_S + C)) \\
\to H^0(\mathcal{I}_Z \cdot \mathcal{O}_S((g - 2)R)).
\]
The first space is zero since
\[
H^0(S, \pi^*\Omega^1_{\mathbb{P}^1}(-B + (k - 2)R)) \cong H^0(S, \mathcal{O}_S(-B + (k - 4)R)),
\]
which is zero by (2.3). Now, since
\[
H^0(S, \mathcal{O}_S((g - 2)R)) \cong H^0(S, \pi^*\mathcal{O}_{\mathbb{P}^1}(g - 2))
\]
the former space can be represented by homogeneous polynomials of degree \( \leq g - 2 \) over \( \mathbb{P}^1 \), while \( H^0(\mathcal{I}_Z \cdot \mathcal{O}_S((g - 2)R)) \) is isomorphic to those polynomials which vanish on the scheme \( Z \). The Riemann-Hurwitz formula shows that there are at least \( g + 2 \) ramification points of the restriction map \( \pi|_C : C \to \mathbb{P}^1 \) (with equality if all are total ramification points); but no polynomial of degree \( \leq g - 2 \) can vanish on \( \geq g + 2 \) points. Therefore
\[
H^0(\mathcal{I}_Z \cdot \mathcal{O}_S((g - 2)R)) = 0, \quad H^0(S, \mathcal{O}^1_{S}(\log C)(2K_S + C)) = 0,
\]
and the proposition is proved.

4. Trigonal curves of genus \( g = 4 \)

(4.1) To finish the proof of the theorem, we show that \( \text{corank}(\Phi_K) = g + 5 = 9 \) for the trigonal curves of genus 4, each of which is the complete intersection of a quadric and a cubic surface in \( \mathbb{P}^3 \). Accordingly we construct the following
diagram involving the Gaussian map $\Phi$ for the divisor $\mathcal{O}_P$, (1) on $\mathbb{P}^3$:

$$
\xymatrix{ 
\Lambda^2 H^0(\mathbb{P}^3, \mathcal{O}_P, (1)) \ar[r]_{\Phi} \ar[d]_{\text{Res}} & H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(2)) \ar[d]_{\gamma} \ar[r]_{\alpha} & H^0(C, \Omega^1_{\mathbb{P}^3}(2)|_C) \ar[d]_{\beta} \\
\Lambda^2 H^0(C, \Omega^1_C) \ar[r]_{\Phi_K} & H^0(C, \Omega^1_C) 
}
$$

By a theorem of Wahl the Gaussian map for any divisor $\mathcal{O}_P(k)$ on $\mathbb{P}^n$ is surjective ($n \geq 1$), so the top map $\Phi$ is surjective [Wa]. The left vertical map $\text{Res}$ is again surjective due to the normality of $\mathbb{P}^3$, and the right map $\gamma$ factors through the space $H^0(C, \Omega^1_{\mathbb{P}^3}(2)|_C)$. We show that $\text{Im}(\alpha) \cap \ker(\beta) = 0$ and hence that the Gaussian-Wahl map has rank equal to the rank of the map $\alpha$ in the diagram above. We write \{\(x_jdx_i - x_i dx_j\) | \(0 \leq i < j \leq 3\)} as a 6-dimensional basis for the space $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(2))$. An element of $\text{Im}(\alpha) \cap \ker(\beta)$ can be expressed as a quadric polynomial $Q$ that vanishes along the curve $C$ such that

$$
dQ \equiv \sum_{0 \leq i < j \leq 3} \lambda_{ij}(x_jdx_i - x_i dx_j) \pmod{I_C},
$$

where $\lambda_{ij} \in \mathbb{C}$. Since $C$ is not contained in any hyperplane of $\mathbb{P}^3$, this means

$$
\frac{\partial Q}{\partial x_i} = \sum_{j > i} \lambda_{ij}x_j - \sum_{j < i} \lambda_{ji}x_i \quad \forall i,
$$

and $2Q = 0$ by Euler's lemma. Therefore, $\text{Im}(\alpha) \cap \ker(\beta) = 0$, $\text{rank}(\Phi_K) = \text{rank}(\alpha) = 6$, and $\text{corank}(\Phi_K) = 9$. □

References


Department of Mathematics and Computer Science, St. John's University, Jamaica, New York 11439

E-mail address: brawnerj@sjuvm.stjohns.edu