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FUNCTIONS ON NONCOMPACT LIE GROUPS WITH POSITIVE FOURIER TRANSFORMS

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ABSTRACT. Let G be a homogeneous group with the graded Lie algebra or a noncompact semisimple Lie group with finite center. We define the Fourier transform $\hat{f}$ of f as a family of operators $\hat{f}(\pi) = \int_G f(x)\pi(x)\,dx$ ($\pi \in \widehat{G}$), and we say that $\hat{f}$ is positive if all $\hat{f}(\pi)$ are positive. Then, we construct an integrable function f on G with positive $\hat{f}$ and the restriction of f to any ball centered at the origin of G is square-integrable, however, f is not square-integrable on G.

1. INTRODUCTION

When G is a compact abelian group, integrable functions f on G with the nonnegative Fourier *coefficients* and being pth ($1 < p \le 2$) power integrable near the identity of G have the Fourier coefficients in l^q ($q = p/(p-1)$). This result was first obtained by N. Wiener in the case of $G = \mathbf{T}$ and $p = 2$ (cf. [3]) and then by Rains [9] and Ash, Rains, and Vági [1] for arbitrary compact abelian groups. When G is a compact semisimple Lie group, an analogous result was obtained by the author and Miyazaki [6]. Furthermore, Nassiet [8] and Blank [2] treated the same problem in the case that G is a compact separable group.

When G is not compact, for example, when $G = \mathbf{R}$, a counterexample was obtained by the author, Onoe, and Tachizawa [7]: there exists an integrable function f on $\mathbf{R}$ with nonnegative Fourier *transform* and the restriction of f to a neighborhood of 0 is square-integrable, however, f is not square-integrable on $\mathbf{R}$. In this paper we shall also give a counterexample when G is a homogeneous group with the graded Lie algebra (see [4, Chapter 1]) and also when G is a noncompact semisimple Lie group with finite center. In the case of a homogeneous group with the graded Lie algebra we can find a one-parameter subgroup $\mathscr{A}$ of G for which $axa^{-1} = x$ for all $a \in \mathscr{A}$ and $x \in G$. Then, regarding $\mathscr{A}$ as $\mathbf{R}$, we can apply the same idea used in [7] to construct the counterexample on G. As compared with [7], our proof is simple and group-theoretical. Especially, the condition (3) in [7] can be replaced by a weaker condition. In the case of a noncompact semisimple Lie group we can take a one-parameter subgroup $\mathscr{A}$ of G as a subgroup of the maximal abelian subgroup A of G. Although $\mathscr{A}$

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does not belong to the center of G, the same idea is still applicable to obtain a counterexample.

2. Homogeneous groups

2.1. **Notation.** Let G be a homogeneous group whose Lie algebra $\mathfrak{g}$ is graded and $|\cdot|: G \to \mathbf{R}_+$ a homogeneous norm of G (see [4, Chapter 1]). $\mathfrak{g}$ is endowed with a vector space decomposition $\mathfrak{g} = \sum_{k=1}^{\infty} V_k$ such that $[V_i, V_j] \subset V_{i+j}$, where all but finitely many V_k's are $\{0\}$. Then if we take $k_0 = \max\{k; V_k \neq \{0\}\}$ and $X \neq 0 \in V_{k_0}$, $\mathscr{A} = \exp(\mathbf{R}X)$ satisfies

$$(*) \qquad axa^{-1} = x \quad \text{for all } a \in \mathscr{A} \text{ and } x \in G.$$

Some examples may be in order: (i) Noncompact abelian groups $\mathbf{R}^n$; (ii) Heisenberg group H_n. The underlying manifold is $\mathbf{C}^n \times \mathbf{R}$ and the multiplication law is given as

$$\begin{aligned}(z_1, \dots, z_n, t)&(z_1', \dots, z_n', t') \\ &= \left(z_1 + z_1', \dots, z_n + z_n', t + t' + 2\Im\left(\sum_{j=1}^{n} z_j \overline{z}_j'\right)\right).\end{aligned}$$

Then $\mathscr{A} = (0, \dots, 0, \mathbf{R})$ satisfies $(*)$; and (iii) The group of all upper triangle matrices $(a_{ij})_{1\le i,j\le n}$ with $a_{jj} = 1$ $(1 \le j \le n)$. Then $\mathscr{A} = \exp(\mathbf{R}E_{1n})$ satisfies $(*)$, where E_{1n} is the matrix with 0 entries but 1 in the $(1, n)$ entry. Let dx be a G-invariant measure on G. We denote the volume of a measurable set S of G by $|S|$ and the L^p-norm $(1 \le p < \infty)$ of a function f on G by $\|f\|_p = (\int_G |f(x)|^p\, dx)^{1/p}$. For any integrable functions f on G we denote the Fourier transform $\hat{f}$ of f as a family of operators $\hat{f}(\pi) = \int_G f(x)\pi(x)\, dx$ $(\pi \in \widehat{G})$. We say that $\hat{f}$ is positive if all $\hat{f}(\pi)$ $(\pi \in \widehat{G})$ are positive operators (see [11, p. 317]), which we denote by $\hat{f}(\pi) \ge 0$. Let $B(r) = \{x \in G; |x| \le r\}$ $(r \in \mathbf{R}_+)$. Then there exists a positive constant D such that

$$(**) \qquad |B(r)| \sim r^D \qquad (r \in \mathbf{R}_+)$$

(see [4, p. 10]). Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathscr{A}$ such that $|a_n| = n$, and let $\{b_n\}_{n\in\mathbf{N}}$ and $\{r_n\}_{n\in\mathbf{N}}$ be sequences in $\mathbf{R}_+$ satisfying

(1) $r_1 < 1/2$, r_n is decreasing, and there exists $L \in \mathbf{R}_+$ such that $r_n \ge 2r_m$ $(m \ge n)$ if and only if $m \ge Ln$,
(2) $\sum_{n=1}^{\infty} b_n |B(r_n)| < \infty$,
(3) for each $M \in \mathbf{R}_+$, $\sum_{n=1}^{\infty} \sum_{m\in\mathbf{N},\, |n-m|\le M} b_n b_m |B(r_n)|^{1/2} |B(r_m)| < \infty$,
(4) for each $N \in \mathbf{R}_+$,

$$\sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l\ge Nn'}^{\infty} b_n b_{n'} b_{n+l} b_{n'+l} |B(r_{n'})| |B(r_{n+l})| |B(r_{n'+l})| = \infty.$$

Example. Let (α, β) be a pair of positive numbers satisfying (i) $\alpha - \beta + 1 < 0$, (ii) $4\alpha - 3\beta + 2 < 0$, and (iii) $4\alpha - 3\beta + 3 \ge 0$. For instance, $\alpha = 3$ and $\beta = 5$. Here we let $b_n = n^\alpha$ and $|B(r_n)| = n^{-\beta}$. Then (1) is obvious and (2) follows from (i). For (3) it is enough to estimate the sum of n over $2M$ and then

$$\sum_{n=2M}^{\infty} \sum_{\substack{m\in\mathbf{N} \\ |n-m|\le M}} n^{\alpha} m^{\alpha} n^{-\beta/2} m^{-\beta} \le c \sum_{n=2M}^{\infty} n^{2\alpha-3\beta/2} < \infty$$

by (ii). (4) follows from (iii) as

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l\ge Nn'}^{\infty} n^{\alpha} n'^{\alpha} (n+l)^{\alpha} (n'+l)^{\alpha} n'^{-\beta} (n+l)^{-\beta} (n'+l)^{-\beta} \\ &\quad \ge \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} n^{\alpha} n'^{\alpha-\beta} \int_{Nn'}^{\infty} (n+x)^{\alpha-\beta} (n'+x)^{\alpha-\beta}\, dx \\ &\quad \ge \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} n^{\alpha} n'^{3\alpha-3\beta+1} \int_{N}^{\infty} (x+1)^{2\alpha-2\beta}\, dx \\ &\quad \ge c \sum_{n=1}^{\infty} n^{\alpha} \int_{n}^{\infty} x^{3\alpha-3\beta+1}\, dx \\ &\quad \ge c \sum_{n=1}^{\infty} n^{4\alpha-3\beta+2} = \infty. \end{aligned}$$

2.2. **Counterexample.** For each measurable set S of G we denote by χ_S the characteristic function of S. Now we define a function g_n $(n \in \mathbf{N})$ on G as $g_n(x) = b_n \mathbf{x}_n(x) = b_n \chi_{B(r_n)}(a_n^{-1}x)$ $(x \in G)$ and put $g = \sum_{n=1}^{\infty} g_n$. Then, $\|g\|_1 = \sum_{n=1}^{\infty} \|g_n\|_1 = \sum_{n=1}^{\infty} b_n |B(r_n)| < \infty$ by (2). Here we let $f = g^{\sim} * g$, where $g^{\sim}(x) = g(x^{-1})$. Then

(5) $\|f\|_1 \le \|g^{\sim}\|_1 \|g\|_1 = \|g\|_1^2 < \infty$ and $\hat{f}(\pi) = \hat{g}(\pi^*)\hat{g}(\pi) \ge 0 \qquad (\pi \in \hat{G})$.

Since $\operatorname{supp}(\mathbf{x}_n^{\sim} * \mathbf{x}_m) = B(r_n) a_{m-n} B(r_m) \subset B(r_1) a_{m-n} B(r_1)$ (see (1)), it follows that $g_n^{\sim} * g_m(x) = 0$ if $x \in B(R)$ $(R \in \mathbf{R}_+)$ and $|m-n| > 2r_1 + R$. Therefore, we can deduce that for each $R \in \mathbf{R}_+$

(6)
$$\begin{aligned} \left(\int_{B(R)} |f(x)|^2\, dx \right)^{1/2} &\le \sum_{n=1}^{\infty} \sum_{\substack{m\in\mathbf{N} \\ |n-m|\le 2r_1+R}} \|g_n^{\sim} * g_m\|_2 \\ &\le \sum_{n=1}^{\infty} \sum_{\substack{m\in\mathbf{N} \\ |n-m|\le 2r_1+R}} \|g_n\|_2 \|g_m\|_1 \\ &= \sum_{n=1}^{\infty} \sum_{\substack{m\in\mathbf{N} \\ |n-m|\le 2r_1+R}} b_n b_m |B(r_n)|^{1/2} |B(r_m)| < \infty \end{aligned}$$

by (3). Finally we show that $\|f\|_2 = \infty$. Since

$$\mathbf{x}_n^{\sim} * \mathbf{x}_m(x) = |a_n B(r_n) x \cap a_m B(r_m)| = |a_{m-n}^{-1} B(r_n) x \cap B(r_m)| \qquad (x \in G),$$

it follows from $(*)$ that if $m \ge n$ and $x \in a_{m-n} B(r_n - r_m)$, then $a_{m-n}^{-1} B(r_n) x \supset B(r_m)$ and thus, $\mathbf{x}_n^{\sim} * \mathbf{x}_m(x) = |B(r_m)|$. On the other hand, (1) implies that there exists $N = L - 1 \in \mathbf{R}_+$ such that $r_n - r_m \ge r_n/2$ if and only if $m \ge (N+1)n$.

Therefore, we can deduce that if $m \geq (N+1)n$, then $\mathbf{x}_n^{\sim} * \mathbf{x}_m(x) = |B(r_m)|$ for $x \in B(r_n/2)$. Since $|B(r)| \sim |B(2r)|$ by $(**)$, it follows from (4) that

$$\begin{aligned} \|f\|_2^2 &= g^{\sim} * g * (g^{\sim} * g)^{\sim}(1) \\ &\geq \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l \geq Nn'}^{\infty} b_n b_{n'} b_{n+l} b_{n'+l} \mathbf{x}_n^{\sim} * \mathbf{x}_{n+l} * \mathbf{x}_{n'}^{\sim} * \mathbf{x}_{n'+l}(1) \\ &\geq \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l \geq Nn'}^{\infty} b_n b_{n'} b_{n+l} b_{n'+l} |B(r_{n'})| |B(r_{n+l})| |B(r_{n'+l})| = \infty, \end{aligned} \tag{7}$$

where the symbol "$\sim$" means that the ratio of the right- and left-hand sides is bounded below and above by positive constants. Then (5)–(7) implies the following

Theorem 1. *Let G be a homogeneous group with graded Lie algebra. Then there exists an integrable function f on G with positive $\hat{f}$ and the restriction of f to any ball centered at the origin of G is square-integrable, however, f is not square-integrable on G.*

3. Semisimple Lie groups

3.1. **Notation.** Let G be a noncompact semisimple Lie group with finite center and $G = KCL(A_+)K$ a Cartan decomposition of G. Let $\sigma: G \to \mathbf{R}_+$ denote the K-bi-invariant function on G defined by $\sigma(x) = d(\overline{1}, \overline{x})$ $(x \in G)$, where d is the Riemannian distance on the symmetric space $X = G/K$, $x \to \overline{x}$ is the natural map of G to X, and 1 is the origin of G (cf. [10]). Let $\mathscr{A} = \{a_t \in A; t \in \mathbf{R}\}$ be a one-parameter subgroup of G for which $\{a_n\}_{n \in \mathbf{N}}$ is a sequence in A_+ such that $\sigma(a_n) = n$. Let dx be a G-invariant measure on G. As in the case of homogeneous groups, we define the volume $|S|$ of a measurable set S of G, the L^p-norm $\|f\|_p$, and the Fourier transform $\hat{f}$ of a function f on G. Let $B(r) = \{x \in G; \sigma(x) \leq r\}$. Then there exists a positive constant D such that

$$|B(r)| \sim r^D \qquad (r < 1) \tag{$***$}$$

(cf. [5, Chapter X]). We fix two sequences $\{b_n\}_{n \in \mathbf{N}}$ and $\{r_n\}_{n \in \mathbf{N}}$ in $\mathbf{R}_+$ satisfying the exactly same conditions (1)–(4).

3.2. **Counterexample.** We define a right K-invariant function g_n $(n \in \mathbf{N})$ as $g_n(x) = b_n \mathbf{x}_n(x) = b_n \chi_{B(r_n)}(a_n^{-1}x)$ $(x \in G)$. We put $g = \sum_{n=1}^{\infty} g_n$ and define a K-bi-invariant function f on G as $f = g^{\sim} * g$. By the same arguments which yield (5) and (6), we see that $f \in L^1(G)$, $\hat{f} \geq 0$, and $f|_{B(R)} \in L^2(G)$ for each $R \in \mathbf{R}_+$. Now we show that $\|f\|_2 = \infty$. Although $(*)$ does not hold for $\mathscr{A}$, it follows that

$$aB(r)a^{-1}K \supset B(r) \quad \text{for all } a \in A \text{ and } r \in \mathbf{R}_+. \tag{$****$}$$

Therefore, if $m \geq n$ and $x = a_{m-n}z \in a_{m-n}B(r_n - r_m)$, we can deduce that $a_{m-n}^{-1}B(r_n)\overline{x} \supset B(r_n)\overline{z} \supset B(r_m)$; and thus, applying the same argument used in the case of homogeneous groups, we can obtain that $\|f\|_2 = \infty$.

Theorem 2. *Let G be a noncompact semisimple Lie group with finite center. Then there exists an integrable K-bi-invariant function f on G with positive $\hat{f}$ and the restriction of f to any ball centered at the origin of G is square-integrable, however, f is not square-integrable on G.*

In the proofs of Theorems 1 and 2 the structure of Lie groups is not essential. Actually, let G be a noncompact separable group and suppose that G has a one-parameter subgroup $\mathscr{A} = \{a_t ; t \in \mathbf{R}\}$ of G and the family of neighborhoods of the identity of G parametrized as $B(r)$ $(0 < r \leq 1)$ satisfying (i) $B(r) \subset B(r')$ if $r < r'$, (ii) $|B(r)| \sim r^D$ for $D > 0$, (iii) $a_n B(1)$ $(n \in \mathbf{N})$ are disjoint, and (iv) $aB(r)a^{-1} \supset B(r)$ for all $a \in \mathscr{A}$ and $0 < r \leq 1$. Then, by the same argument used in the proof of Theorem 2, we can construct the counterexample for G.

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