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CORRECTION TO "ON THE DESKINS INDEX COMPLEX OF A MAXIMAL SUBGROUP OF A FINITE GROUP"

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This note is to correct a mistake in [1]. So, the notation, definitions, and references are those of that paper.

If M is maximal subgroup of a finite group G and H/K is a chief factor of G supplemented by M we cannot say in general that $H \in I(M)$. For example, if G is the dihedral group of order 30, M a maximal subgroup of G isomorphic to $\mathrm{Sym}(3)$, and $H = \mathrm{Soc}(G)$, we have that $H \notin I(M)$ since $k(H) = H$. (Compare this with the last paragraph of page 236 in [5].) This motivates that Proposition 1 in our paper does not hold. Changing that by Proposition 1* below, we prove below that the five theorems of the paper remain true.

Remark. If there exists $C \in I(M)$ such that C is a normal subgroup of G, then C is a maximal completion of M in G and $C/k(C)$ is a chief factor of G which is isomorphic to any minimal normal subgroup of G/M_G; in particular, if M supplements the chief factor A/B of G, then $C/k(C) \cong A/B$.

Proof. Since C is a normal subgroup of G, there is no completion $C_1 \in I(M)$ such that $C \leq C_1$ and then C is a maximal completion for M in G. It is clear that $C/k(C)$ is a chief factor of G and $k(C) = M_G \cap C$, where M_G denotes $\mathrm{core}_G(M)$. Now G factorizes as $G = MC$ and in the primitive group G/M_G the subgroup CM_G/M_G is a minimal normal subgroup which is clearly isomorphic to $C/k(C)$. Finally recall that in case the primitive group has two minimal normal subgroups, both are isomorphic.

Proposition 1*. *Let $\mathscr{H}$ be a homomorph which is closed under taking normal subgroups. Let G be a group, N a normal subgroup of G, and M a maximal subgroup of G such that $N \leq M$.*

(i) *If C is a maximal completion of M in G such that $C/k(C) \in \mathscr{H}$ and $N \nleq C$, then there exists a normal subgroup A of G such that $AN/N \in I(M/N)$ and $(AN/N)/k(AN/N) \in \mathscr{H}$.*

(ii) *If $C \in S(M)$ such that $C/k(C) \in \mathscr{H}$, then there exists $D/N \in S(M/N)$ such that $(D/N)/k(D/N) \in \mathscr{H}$.*

(iii) *If M has a maximal completion C in G with $C/k(C)$ abelian, then M/N has a maximal completion D/N in G/N with $(D/N)/k(D/N)$ abelian.*

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Proof. (i) Denote by $\mathscr{S}$ the set of all normal subgroups K of G such that $K \leq k(CN)$ and $K \nleq M$. Since $C < CN$, $CN \notin I(M)$ and then $k(CN) \nleq M$ and therefore $\mathscr{S} \neq \varnothing$. Let A be a minimal element of $\mathscr{S}$; it is easy to prove that A is the required subgroup.

(ii) If $N \leq C$, then $C/N \in S(M/N)$ and $k(C/N) = k(C)/N$. So, assume that $N \nleq C$; if $k(CN) \leq M$, then we put $D = CN$ and D/N has the required properties; otherwise argue as in part (i).

(iii) If M possesses a maximal completion C in G such that $C/k(C)$ is abelian, then, by [5, Theorem 2], there exists a normal completion B of M in G with $B/k(B)$ abelian. By part (i) if $N \nleq B$ we are done. So assume that $N \leq B$; in this case $k(B)/N = k(B/N)$ and B/N is the required completion. □

The five following theorems are proved by induction. In all cases we have to prove that a group G satisfying some properties belongs to some particular classes of groups, which are saturated formations. We consider a minimal normal subgroup N of G and show that the group G/N satisfies the same properties to conclude by minimality of G that G/N is in the required class. This was done by means of Proposition 1. So we have replaced those arguments by some new ones using Proposition 1*.

Theorem 1. *A group G is supersoluble if and only if for each maximal subgroup M of G, the family $S(M)$ contains an element C with $C/k(C)$ cyclic.*

Proof. Assume that for each maximal subgroup M of G, the family $S(M)$ contains an element C with $C/k(C)$ cyclic. We see that G is supersoluble by induction on $|G|$. Let N be a minimal normal subgroup of G. If M is a maximal subgroup of G and $N \leq M$, then there exists $C \in S(M)$ such that $C/k(C)$ is cyclic. Now apply Proposition 1*(ii) to deduce the existence of $D/N \in S(M/N)$ such $(D/N)/k(D/N)$ is cyclic. By induction, G/N is supersoluble.

Therefore we can assume that G is a monolithic primitive group.

The remainder of the proof is correct. □

Theorem 2. *Assume that each maximal subgroup of M of a group G has a maximal completion $C \in I(M)$ with $C/k(C)$ nilpotent. Then every composition factor of G is in $\mathscr{X}$, where*

$$\mathscr{X} = \{C_p : p \in \mathbf{P}\} \cup \{\mathrm{PSL}(2, q) : q = 2^n \pm 1,\, q > 5\}.$$

Proof. Denote by $E_{\mathscr{X}}$ the saturated formation composed of all groups whose composition factors are in $\mathscr{X}$. Let G be a minimal counterexample to the theorem. We deduce that G is a monolithic primitive group whose socle N is not in $E_{\mathscr{X}}$ and $G/N \in E_{\mathscr{X}}$.

Consider A/B a nonabelian chief factor of G/N and take M/N a maximal subgroup of G/N supplementing A/B. Denote by $\mathscr{S}$ the nonempty set of maximal completions C of M in G such that $C/k(C)$ is nilpotent. We have that $\mathscr{S} \neq \{G\}$, and by Proposition 1*(i) we have $N \leq C$ for all $C \in \mathscr{S}$.

Consider $C \in \mathscr{S}$ and take $T \leq G$ such that C is a maximal subgroup of T. By Baumann's result $T/k(C) \in E_{\mathscr{X}}$. By maximality of C, we have that $k(T) \nleq M$ and the set $\mathscr{A}$ of all normal subgroups K of G such that $k(C) \leq K \leq K(T)$ and $K \nleq M$ is nonempty. Let K be a minimal element in $\mathscr{A}$. Then $K/k(C) \in I(M/k(C))$, and by the preliminary remark $(K/k(C))/k(K/k(C))$ is a chief factor of G/N and is isomorphic to A/B.

On the other hand $(K/k(C))/k(K/k(C))$ is in $\langle Q, S_n\rangle(T/k(C)) \subseteq E_{\mathscr{X}}$. This implies that A/B is of $\mathscr{X}$-type.

Therefore for any minimal normal subgroup N of G we have $G/N \in E_{\mathscr{X}}$. Since $E_{\mathscr{X}}$ is a saturated formation, we deduce that G is a monolithic primitive group. □

Theorem 3. *Let π be a set of primes. Suppose that G is a group with a maximal subgroup H which is π-soluble. Assume that for each completion $C \in I(H)$ such that $C \cap H \neq 1$, the factor $C/k(C)$ is π-soluble. Then G is π-soluble.*

Proof. Let G be a minimal counterexample to the theorem. Let N be a minimal normal subgroup of G. We have to prove that G/N is π-soluble.

We can assume $N \leq H$. Let $C/N \in I(H/N)$ with $N < C \cap H$. We can assume that $C \notin I(H)$. This means that $k(C) \nleq H$. So there exists a normal completion A of H in G such that $A < k(C)$, $C = AN$, and $k(C/N) = k(A)N/N$. Clearly $N \cap A = 1$ and $A \cap H \neq 1$. By hypothesis $A/k(A)$ is π-soluble and then $(C/N)/k(C/N)$ is π-soluble. Therefore the hypotheses hold in G/N and by minimality of G we deduce that G/N is π-soluble. □

Theorem 4. *Let G be a group, and consider the family $\mathscr{J}(G) = \{M: M$ is a c-maximal subgroup of G and M has no maximal completion C with $C/k(C)$ abelian$\}$. If $S(G) = \bigcap\{M: M \in \mathscr{J}(G)\}$, then $S(G)$ is the soluble radical of G.*

We prove that $S(G)$ is soluble by induction on $|G|$. So if N is a minimal normal subgroup of G, then $S(G/N)$ is soluble. By Proposition 1*(iii), if $M/N \in \mathscr{J}(G/N)$, then $M \in \mathscr{J}(G)$. Then $S(G)N/N \leq S(G/N)$ for each minimal normal subgroup N of G. Therefore we can assume that G is a monolithic primitive group.

The remainder of the proof is correct. □

Theorem 5. *Let G be a group and p a prime. Consider the family $\mathscr{A}(G) = \{M: M$ is a maximal subgroup of G and M has no maximal completion C with $C/k(C)$ abelian$\}$. If each $M \in \mathscr{A}(G)$ is p-nilpotent and if, in addition, either*

(i) *the Sylow p-subgroups of G are abelian, or*
(ii) *p is an odd prime,*

then G is p-soluble.

Proof. Let N be a minimal normal subgroup of G. By Proposition 1*(iii), if $M/N \in \mathscr{A}(G/N)$, then $M \in \mathscr{A}(G)$. Hence every element of $\mathscr{A}(G/N)$ is p-nilpotent. By induction, G/N is p-soluble in both cases. Therefore we may assume that G is a monolithic primitive group.

The remainder of the proof is correct. □

References

1. A. Ballester-Bolinches and Luis M. Ezquerro, *On the Deskins index complex of a maximal subgroup of a finite group*, Proc. Amer. Math. Soc. **114** (1992), 325–330.

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