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NONNEGATIVE SOLUTIONS OF THE RADIAL LAPLACIAN WITH NONLINEARITY THAT CHANGES SIGN

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ABSTRACT. We find a solution to the radial Laplacian equation $y'' + \frac{N-1}{x}y' + \lambda a(x)f(y) = 0$, $y'(0) = y(1) = 0$ when a may change sign and is "sufficiently positive". The function f is qualitatively like e^y, and we conclude solutions for $0 \le \lambda \le \lambda_0$.

1. INTRODUCTION

A variety of papers have looked at boundary value problems of the form

$$Ly = \lambda a(x)f(y), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

or

$$y'(0) = y(1) = 0 \tag{2'}$$

in which $f \ge 0$ and $a \ge 0$, and L is some differential operator. We have discussed this problem in [1], [2], [3], and [4] as well as other papers cited there. In each case we have endeavored to show the existence of solutions by constructive means, that is, have some sort of iteration converge. If $f(0) = 0$ it is especially difficult to get a numerical procedure to converge to a positive solution. In this paper, we continue this study in which we allow, for the first time, the coefficient function a to change sign.

To be specific we study the problem (1) and (2′) where $Ly = -[y'' + \frac{N-1}{x}y']$ for $N \ge 2$. Obvious modifications in our arguments would also suffice for the problem (1) and (2) with $Ly = -y''$.

We are considering finding a positive solution ($y > 0$ in $[0, 1)$) of

$$y'' + \frac{N-1}{x}y' + \lambda a(x)f(y) = 0 \tag{3}$$

with

$$y'(0) = y(1) = 0 \tag{4}$$

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where λ is a positive parameter;

(5) $$a(t) = a_+(t) - a_-(t) \quad \text{with } a_\pm(t) \geq 0 \text{ and } a \in L_1(0, 1);$$

and

(6) f is continuous, positive, and nondecreasing on $[0, \infty)$.

It is well known that the Green's function $G(x, t)$ for (3) and (4) is non-negative when we write it as

$$(x^{N-1}y')' + \lambda x^{N-1}a(x)y(x) = 0.$$

Moreover it is easy to see that a solution to (3) and (4) is a fixed point of the operator

(7) $$T\Phi(x) \equiv \lambda \int_0^1 G(x, t)t^{N-1}a(t)f(\Phi(t))\,dt$$

whose domain will be $C^+(0, 1)$, the cone of continuous non-negative functions. The operator T does not leave this cone invariant since we allow $a(x)$ to change sign.

In order to set up a convergent iteration we will use the following proposition.

Proposition. *Assume* (5) *and* (6). *Let* $A = \{x|a(x) \geq 0\}$ *and* $B = \{x|a(x) < 0\}$. *Suppose we have bounded measurable functions* ϕ_0, ψ_0 *on* $[0, 1]$ *such that they satisfy*

(i) $$0 \leq \phi \leq \psi \quad \text{on } A, \qquad 0 \leq \psi \leq \phi \quad \text{on } B;$$

(ii) $$T\psi \leq \psi \quad \text{on } A, \qquad T\psi \leq \phi \quad \text{on } B;$$

and

(iii) $$T\phi \geq \phi \quad \text{on } A, \qquad T\phi \geq \psi \quad \text{on } B.$$

Define

(iv) $$\phi_1(x) = \begin{cases} T\phi_0(x) & \text{on } A, \\ T\psi_0(x) & \text{on } B; \end{cases} \qquad \psi_1(x) = \begin{cases} T\psi_0(x) & \text{on } A, \\ T\phi_0(x) & \text{on } B. \end{cases}$$

Then ϕ_1 *and* ψ_1 *also satisfy* (i), (ii), *and* (iii).

Proof. We note that the operator T can be written as

(8) $$T\phi(x) = \lambda \int_A G(x, t)t^{N-1}a_+(t)f(\phi(t))\,dt - \lambda \int_B G(x, t)t^{N-1}a_-(t)f(\phi(t))\,dt.$$

For convenience we write

(9) $$T\phi(x) = T_1\phi(x) - T_2\phi(x)$$

where $T_i\phi$, $i = 1, 2$, are both monotone in the sense that $\phi \leq \psi$ implies $T_i\phi \leq T_i\psi$. T_1 acts on $C^+([0, 1] \cap A)$ and T_2 on $C^+([0, 1] \cap B)$. Note that (i) implies that

(10) $$T\phi_0 = T_1\phi_0 - T_2\phi_0 \leq T_1\psi_0 - T_2\psi_0 = T\psi_0.$$

This immediately implies that ϕ_1 and ψ_1 satisfy (i). The condition (ii) for ψ_1 is that on A, $T\psi_1 \le \psi_1 = T\psi_0$ and on B, $T\psi_1 \le \phi_1 = T\psi_0$, i.e., $T\psi_1 \le T\psi_0$. But using (ii) and (iii)

$$T\psi_1 = T_1(T\psi_0) - T_2(T\phi_0) \le T_1(\psi_0) - T_2(\psi_0) = T\psi_0. \tag{11}$$

Similarly condition (iii) for ϕ_1 is that $T\phi_1 \ge \phi_1 = T\phi_0$ on A and $T\phi_1 \ge \psi_1 = T\psi_0$ on B. Now

$$T\phi_1 = T_1(T\phi_0) - T_2(T\psi_0) \ge T_1(\phi_0) - T_2(\phi_0) = T\phi_0 \tag{12}$$

gives (iii) for ϕ_1.

Theorem 1. *Assume* (5) *and* (6), *and suppose there are functions* ϕ_0, ψ_0 *that satisfy* (i), (ii), *and* (iii). *Then the problem* (3)–(4) *has a solution.*

Proof. We define

$$\phi_{n+1} = \begin{cases} T\phi_n & \text{on } A, \\ T\psi_n & \text{on } B; \end{cases} \qquad \psi_{n+1} = \begin{cases} T\psi_n & \text{on } A, \\ T\phi_n & \text{on } B. \end{cases} \tag{13}$$

By the proposition and induction, (ϕ_n, ψ_n) satisfies (i), (ii), and (iii) and hence (10), (11), and (12), i.e.,

$$0 \le T\phi_n \le T\phi_{n+1} \le T\psi_{n+1} \le T\psi_n \le T\psi_0.$$

Thus $T\phi_n \uparrow \overline{\phi}$ and $T\psi_n \downarrow \overline{\psi}$ pointwise and $\overline{\phi} \le \overline{\psi}$. Since

$$T\phi_{n+1} = T_1(T\phi_n) - T_2(T\psi_n),$$

we may apply Lebesgue's dominated convergence theorem to have

$$\overline{\phi} = T_1(\overline{\phi}) - T_2(\overline{\psi}) \tag{14}$$

so that $\overline{\phi}$ is continuous on $[0, 1]$. Similarly $\overline{\psi}$ is continuous on $[0, 1]$, and

$$\overline{\psi} = T_1(\overline{\psi}) - T_2(\overline{\phi}). \tag{15}$$

From (13) we have $\phi_{n+1} \to \overline{\phi}$ on A by definition of $\overline{\phi}$. On the other hand $\phi_{n+1} = T\phi_n$ on A, so $\phi_{n+1} \to T\overline{\phi}$. Thus $\overline{\phi} = T\overline{\phi}$ on A. On B we have $\psi_{n+1} = T\phi_n \to \overline{\phi}$ by definition of $\overline{\phi}$. But $\phi_{n+1} = T\psi_n$ on B, so $\phi_{n+1} \to \overline{\psi}$ on B. Thus on B, $T\overline{\psi} = \overline{\phi}$. In a similar way we have

$$\overline{\phi} = \begin{cases} T\overline{\phi} & \text{on } A, \\ T\overline{\psi} & \text{on } B; \end{cases} \qquad \overline{\psi} = \begin{cases} T\overline{\psi} & \text{on } A, \\ T\overline{\phi} & \text{on } B, \end{cases}$$

and $\overline{\phi}$ and $\overline{\psi}$ are fixed points of T^2.

Now consider the convex region in $C[0, 1]$ defined by $C = \{g(x) | \overline{\phi}(x) \le g(x) \le \overline{\psi}(x),\ x \in [0, 1]\}$. This is invariant under T for

$$Tg = T_1(g) - T_2(g) \le T_1(\overline{\psi}) - T_2(\overline{\phi}) = \overline{\psi}$$

by (15). Similarly

$$Tg = T_1(g) - T_2(g) \ge T_1(\overline{\phi}) - T_2(\overline{\psi}) = \overline{\phi}.$$

Now $G(x, t)t^{N-1}$ is continuous on $[0, 1] \times [0, 1]$, so $a \in L_1(0, 1)$ implies that $\{(Tg) | g \in C\}$ is uniformly equicontinuous, so T restricted to C is a compact operator. By Schauder's fixed point theorem T has a fixed point.

Comment. The order interval $\{g(x)|T\phi_0 \leq g \leq T\psi_0\}$ already is invariant and one could use Schauder's theorem directly. But the iteration improves the estimates. In general one might expect the two functions $\overline{\phi}$ and $\overline{\psi}$ to be the same. So one could in effect construct the solution numerically.

Observation. If $f \geq 0$ and nonincreasing, then one takes (i) and the following modified versions:

(ii)′ $$T\phi \leq \psi \quad \text{on } A, \qquad T\phi \leq \phi \quad \text{on } B;$$

(iii)′ $$T\psi \geq \phi \quad \text{on } A, \qquad T\psi \geq \psi \quad \text{on } B;$$

and

(iv)′ $$\phi_1 = \begin{cases} T\psi_0 & \text{on } A, \\ T\phi_0 & \text{on } B; \end{cases} \qquad \psi_1 = \begin{cases} T\phi_0 & \text{on } A, \\ T\psi_0 & \text{on } B. \end{cases}$$

One can prove the theorem of existence where now $T\phi \geq T\psi$, $T\phi\downarrow$, and $T\psi\uparrow$. The details are the same.

3. An example

To construct an example where the previous analysis applies, we will make the following assumption.

(H): There is an $\varepsilon > 0$ so that

$$\int_0^t x^{N-1}a_+(x)\,dx \geq (1+\varepsilon)\int_0^t x^{N-1}a_-(x)\,dx \quad \text{for all } t \in [0, 1]. \tag{16}$$

Comment. H means a is sufficiently positive near 0.

We seek ϕ_0 and ψ_0 so that (i), (ii), and (iii) are satisfied. Let $\phi_0(x) = \alpha$ on B and $\phi_0(x) = 0$ on A with $\psi_0(x) = \alpha$ on A and $\psi_0(x) = 0$ on B. Then (i) is satisfied if $\alpha \geq 0$. Now the condition (ii) is

$$T\psi_0 = T_1(\alpha) - T_2(0) \leq \alpha \quad \text{on } [0, 1]$$

while (iii) is

$$T\phi_0 = T_1(0) - T_2(\alpha) \geq 0 \quad \text{on } [0, 1].$$

Letting $z_\pm(x) \equiv \int_0^1 G(x, t)t^{N-1}a_\pm(t)\,dt$ these conditions become

$$\lambda[z_+(x)f(\alpha) - z_-(x)f(0)] \leq \alpha \tag{17}$$

and

$$\lambda[z_+(x)f(0) - z_-(x)f(\alpha)] \geq 0. \tag{18}$$

We consider (18) first. Define $z(x) = z_+(x) - (1+\varepsilon)z_-(x)$. Then z is a solution of

$$z'' + \frac{N-1}{x}z + (a_+ - (1+\varepsilon)a_-) = 0, \qquad z'(0) = z(1) = 0.$$

It follows that

$$(z'x^{N-1})' = -x^{N-1}(a_+(x) - (1+\varepsilon)a_-(x))$$

and

$$z'(t)t^{N-1} = -\int_0^t x^{N-1}(a_+(x) - (1+\varepsilon)a_-(x)]\,dx \leq 0$$

by (H). Thus z is decreasing and therefore is non-negative, i.e., $z_+(x) \geq (1+\varepsilon)z_-(x)$ on $[0, 1]$. So (18) is satisfied if

$$f(\alpha) \leq (1+\varepsilon)f(0). \tag{19}$$

We select such an α and argue that (17) can now be satisfied for small λ. To prove this we will give an explicit estimate. Now as above

$$z_+(x) - z_-(x) = \int_0^1 G(x, t)t^{N-1}a(t)\,dt$$

and the right-hand side is a decreasing function, so

$$z_+(x) \leq z_-(x) + \int_0^1 G(0, t)t^{N-1}a(t)\,dt.$$

This last integral is

$$\frac{1}{N-2}\int_0^1 t(1-t^{N-2})a(t)\,dt \equiv \beta \qquad \left(\int_0^1 \ln t a(t)\,dt \equiv \beta \text{ if } N=2\right).$$

Hence

$$\begin{aligned} f(\alpha)z_+(x) - f(0)z_-(x) &\leq [f(\alpha) - f(0)]z_-(x) + f(\alpha)\beta \\ &\leq [f(\alpha) - f(0)]z_-(0) + f(\alpha)\beta \end{aligned}$$

since z_- is decreasing (as above for z).

This in turn is dominated by (see (19))

$$\varepsilon f(0)z_-(0) + (1+\varepsilon)f(0)\beta.$$

So (17) is satisfied if

$$\lambda \leq \frac{\alpha}{f(0)[\varepsilon z_-(0) + (1+\varepsilon)\beta]} \tag{20}$$

the denominator is explicitly $(N \geq 3)$

$$\begin{aligned} &f(0)\left[\varepsilon\int_0^1 \frac{(t-t^{N-1})}{N-2}a_-(t)\,dt + (1+\varepsilon)\int_0^1 \frac{(t-t^{N-1})}{N-2}a(t)\,dt\right] \\ &\quad = f(0)\left[\int_0^1 \frac{(t-t^{N-1})}{N-2}a(t)\,dt + \varepsilon\int_0^1 \frac{(t-t^{N-1})}{N-2}a_+(t)\,dt\right]. \end{aligned}$$

Theorem. *If* $f(0) > 0$, f *is non-decreasing, and* a *is measurable and in* $L^1(0, 1)$ *such that* (H) *is satisfied, then problem* (1)-(2′) *has a solution for* $0 \leq \lambda \leq \lambda_0$ *where* $(N \geq 3)$

$$\lambda_0 = \frac{(N-2)\alpha}{f(0)\int_0^1(t-t^{N-1})a(t)\,dt + \varepsilon\int_0^1(t-t^{N-1})a_+(t)\,dt}$$

if $f(\alpha) \leq f(0)(1+\varepsilon)$. *For* $N=2$,

$$\lambda_0 = \frac{-\alpha}{f(0)\int_0^1 t\ln t a(t)\,dt + \varepsilon\int_0^1 t\ln t a_+(t)\,dt}.$$

For specific examples, one can take $f(\alpha) = f(0)(1+\varepsilon)$. For example, if $f(y) = e^y$, $\alpha = \ln(1+\varepsilon)$, and $f(y) = 1 + y^P$, then $\alpha = \varepsilon^{1/p}$. For an example of (H) take

$$a(t) = \begin{cases} A, & 0 \le t < \frac{1}{2}, \\ -B, & \frac{1}{2} < t \le 1, \end{cases}$$

where $0 < B(2^N - 1) < A$. One can easily verify that $1 + \varepsilon = \frac{A}{B(2^N-1)}$ will work for (H). In fact if $a(t)$ is decreasing on $[0, 1]$ with $a(x_0) = 0$, then $\int_0^t x^{N-1} a_+(x) dt \ge 0$ on $[0, x_0]$ and equal to $\int_0^{x_0} x^{N-1} a_+(x)\, dx$ on $[x_0, 1]$. Since $\int_0^t x^{N-1} a_-(x)\, dx = 0$ on $[0, x_0]$, (H) is satisfied if $\int_0^{x_0} x^{N-1} a_+(x)\, dx \ge (1+\varepsilon) \int_0^t x^{N-1} a_-(x)\, dx$ on $[x_0, 1]$. Since the right-hand side is increasing, the condition is

$$(1+\varepsilon) \le \frac{\int_0^1 x^{N-1} a_+(x)\, dx}{\int_0^1 x^{N-1} a_-(x)\, dx}.$$

We have given a sufficient condition for small eigenvalues of the problem (1)–(2′) which involve a_+ being sufficiently positive, i.e., the condition (H). Some such hypothesis is necessary, but we believe that λ small is a correct condition.

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