BEST CONSTANTS FOR TWO NONCONVOLUTION INEQUALITIES

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ABSTRACT. The norm of the operator which averages \(|f|\) in \(L^p(\mathbb{R}^n)\) over balls of radius \(\delta |x|\) centered at either 0 or \(x\) is obtained as a function of \(n\), \(p\) and \(\delta\). Both inequalities proved are \(n\)-dimensional analogues of a classical inequality of Hardy in \(\mathbb{R}^1\). Finally, a lower bound for the operator norm of the Hardy-Littlewood maximal function on \(L^p(\mathbb{R}^n)\) is given.

0. Introduction

A classical result of Hardy [HLP] states that if \(f\) is in \(L^p(\mathbb{R})\) for \(p > 1\), then

\[
\left( \int_0^\infty \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right)^p \, dx \right)^{1/p} \leq \frac{p}{p - 1} \left( \int_0^\infty |f(t)|^p \, dt \right)^{1/p},
\]

and the constant \(p/(p - 1)\) is the best possible. By considering two-sided averages of \(f\) instead of one-sided, (0.1) can be equivalently formulated as:

\[
\left( \int_{-\infty}^\infty \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} |f(t)| \, dt \right)^p \, dx \right)^{1/p} \leq \frac{p}{p - 1} \left( \int_{-\infty}^\infty |f(t)|^p \, dt \right)^{1/p}.
\]

We denote by \(D(a, R)\) the ball of radius \(R\) in \(\mathbb{R}^n\) centered at \(a\). Let \((Tf)(x)\) be the average of \(|f| \in L^p(\mathbb{R}^n)\) over the ball \(D(0, |x|)\). The analogue of (0.2) for \(\mathbb{R}^n\) is the inequality:

\[
\|Tf\|_{L^p} \leq C_p(n) \|f\|_{L^p}
\]

for some constant \(C_p(n)\) which depends a priori on \(p\) and \(n\). Our first result is that the best constant \(C_p(n)\) which satisfies (0.3) for all \(f \in L^p(\mathbb{R}^n)\) is \(p' = p/(p - 1)\), the same constant as in dimension one. Another version of Hardy’s inequality in \(\mathbb{R}^n\) with the best possible constant can be found in [F].

Next we consider a similar problem. An equivalent formulation of (0.1) and (0.2) is

\[
\left( \int_{-\infty}^\infty \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} |f(t)| \, dt \right)^p \, dx \right)^{1/p} \leq \frac{p}{2^{1/p}(p - 1)} \left( \int_{-\infty}^\infty |f(t)|^p \, dt \right)^{1/p},
\]

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where \( f \) is in \( L^p(\mathbb{R}^n) \). Let \( (Sf)(x) \) be the average of \( |f| \in L^p(\mathbb{R}^n) \) over the ball \( D(x, |x|) \). We compute the operator norm \( c_{p,n} \) of \( S \) on \( L^p(\mathbb{R}^n) \) as a function of \( n \) and \( p \). The precise value of the constant \( c_{p,n} \) is given in Theorem 2.

In section 3 a lower bound for the operator norm of the Hardy-Littlewood maximal function on \( L^p(\mathbb{R}^n) \) is given. Finally, in section 4 the norm on \( L^p(\mathbb{R}^n) \) of the operator which averages \( f \) over the ball of radius \( \delta|x| \) centered at either 0 or \( |x| \) is given as a function of \( \delta, p, \) and \( n, \) for any \( \delta > 0 \).

Throughout this note, \( \omega_{n-1} \) will denote the area of the unit sphere \( S^{n-1} \) and \( v_n \) the volume of the unit ball in \( \mathbb{R}^n \).

### 1. Hardy's inequality on \( \mathbb{R}^n \)

In this section we will prove inequality (0.3) with constant \( C_p(n) = p' = p/(p - 1) \). We denote by \( |A| \) the Lebesgue measure of the set \( A \) and by \( \chi_A \) its characteristic function.

**Theorem 1.** Let \( f \in L^p(\mathbb{R}^n) \), where \( 1 < p < \infty \). The following inequality holds:

\[
(1.1) \quad \left( \int_{\mathbb{R}^n} \left( \frac{1}{|D(0, |x|)|} \int_{D(0, |x|)} |f(y)|^p \, dy \right)^{1/p} \, dx \right)^{1/p} \leq \frac{p}{p - 1} \left( \int_{\mathbb{R}^n} |f(y)|^p \, dy \right)^{1/p},
\]

and the constant \( p' = p/(p - 1) \) is the best possible.

**Proof.** Fix \( f \in L^p(\mathbb{R}^n) \). Without loss of generality, assume that \( f \) is non-negative and continuous. Let \( \mathbb{R}^+ \) denote the multiplicative group of positive real numbers with Haar measure \( dt \). The function \( t^{n/p'} \chi_{[0,1]} \) is in \( L^1(\mathbb{R}^+, \frac{dt}{t}) \) with norm \( p'/n \). For a fixed \( \theta \) in the unit sphere \( S^{n-1} \), the function \( t \to f(t\theta)t^{p/n} \) is in \( L^p(\mathbb{R}^+, \frac{dt}{t}) \). The group inequality \( \|g * K\|_{L^p} \leq \|g\|_{L^p} \|K\|_{L^1} \) gives:

\[
(1.2) \quad \int_0^\infty \left( \int_0^1 f(r\theta)(rt)^{\frac{n}{p'}} \frac{dt}{t} \right) \frac{dr}{r} \leq \left( \int_0^\infty (f(r\theta)r^{n/p}) \frac{dr}{r} \right) \left( \frac{p'}{n} \right)^p.
\]

Note that equality holds in (1.2) if and only if equality holds in \( \|g * K\|_{L^p} \leq \|g\|_{L^p} \|K\|_{L^1} \). This happens in the limit by the sequence \( g_{\epsilon,N} = \chi_{[\epsilon,N]} \). Since \( g(t) = f(t\theta)t^{n/p} \), we conclude that equality is attained in (1.2) in the limit by the sequence

\[
(1.3) \quad f_{\epsilon,N}(t\theta) = t^{-n/p} \chi_{\epsilon \leq t \leq N} \quad \text{as} \quad \epsilon \to 0 \quad \text{and} \quad N \to \infty.
\]

Note that \( Tf \) is a radial function. Expressing all integrals in polar coordinates, we reduce (1.1) to a convolution inequality on the multiplicative group \( \mathbb{R}^+ \). We have

\[
(1.4) \quad \|Tf\|_{L^p(\mathbb{R}^n)}^p = \omega_{n-1} \int_0^\infty \left( \frac{1}{v_n r^n} \int_0^r \int_{\theta \in S^{n-1}} f(t\theta)t^{n-1} \, d\theta \, dt \right)^p \, dr \, d\theta.
\]

We apply Hölder's inequality with exponents \( \frac{1}{p'} + \frac{1}{p'} = 1 \) to the functions \( 1 \) and \( \theta \to \int_0^1 f(r\theta)(rt)^{n/p} \frac{dt}{t} \) and then to Fubini's theorem to interchange the integrals in \( \theta \) and \( r \). We obtain that (1.4) is bounded above by

\[
(1.5) \quad \frac{\omega_{n-1}}{v_n} \omega_{n-1} \int_{S^{n-1}} \int_0^\infty \left( \int_0^1 f(r\theta)(rt)^{\frac{n}{p'}} \frac{dt}{t} \right) \frac{dr}{r} \, d\theta.
\]
Note that if \( f \) is a radial function, then (1.4) and (1.5) are identical. We now apply (1.2) to majorize (1.5) by

\[
\frac{\omega_{n-1}}{v_n^p} \left( \frac{p'}{n} \right)^p \int_{S^{n-1}} \int_0^\infty f(r\theta)^p r^n \frac{dr}{r} d\theta = \left( \frac{p}{p-1} \right)^p \|f\|_{L^p(\mathbb{R}^n)}^p
\]

using the fact that \( \omega_{n-1} = nv_n \). We have now obtained the inequality \( \|Tf\|_{L^p} \leq p'\|f\|_{L^p} \). Equality holds when the family of functions (1.3) is radial. Therefore, the extremal family for inequality (1.1) is \(|x|^{-n/p} \chi_{0 \leq |x| \leq N} \), as \( \epsilon \to 0 \) and \( N \to \infty \).

2. A VARIANT OF HARDY'S INEQUALITY ON \( \mathbb{R}^n \)

The derivation of the \( n \)-dimensional analogue of (0.4) is more subtle. Let \( B(s, t) \) denote the usual beta-function \( \int_0^1 x^s (1 - x)^t dx \). Our second result is

**Theorem 2.** Let \( 1 < p < \infty \) and \( c_{p,n} = p' \frac{\alpha_{n-1}}{\alpha_{n-2}} 2^{\frac{n}{2} - 1} B\left( \frac{n}{2}, \frac{n-2}{2} \right) \). The following inequality holds for all \( f \) in \( L^p(\mathbb{R}^n) \):

\[
\left( \int_{\mathbb{R}^n} \left( \frac{1}{|D(x, |x||)} \int_{D(x, |x|)} |f(y)| dy \right)^p dx \right)^{1/p} \leq c_{p,n} \left( \int_{\mathbb{R}^n} |f(y)|^p dy \right)^{1/p}
\]

and the constant \( c_{p,n} \) is the best possible.

**Proof.** We use duality. Fix \( f \) and \( g \) positive and continuous with \( \|f\|_{L^p(\mathbb{R}^n)} \leq 1 \) and \( \|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1 \). We will show that \( \int g(x)(Sf)(x) dx \leq c_{p,n} \). We express both \( g \) and \( Sf \) in polar coordinates by writing \( x = r\phi \) and \( y = t\theta \). The relation \( |x - y| \leq |x| \) is equivalent to \( \theta \cdot \phi \geq t/2r \). We obtain

\[
\int_{\mathbb{R}^n} g(x)(Sf)(x) dx
\]

where \( G(\phi) = \int_{\mathbb{R}^n} G(\phi)F(\theta) \left( \int_0^{\theta \cdot \phi} t^{\frac{p'}{2}} \frac{dt}{t} \right)^{1/p} d\phi d\theta \).

The bracketed expression in (2.2) is the \( L^p \) norm of the group \( (\mathbb{R}^+, t^{\frac{p'}{2}}) \) convolution of the function \( t \to f(t\theta) t^{\frac{p'}{2}} \) with the kernel \( \chi_{[0, \theta \cdot \phi]}(t) t^{\frac{p'}{2}} \) at \( 2r \). We therefore estimate (2.2) by

\[
\frac{2^{\frac{p'}{2}}}{v_n} \int_{(S^{n-1})^2} G(\phi)F(\theta) \left( \int_0^{\theta \cdot \phi} t^{\frac{p'}{2}} \frac{dt}{t} \right)^{1/p} d\phi d\theta \,
\]

where \( F(\theta) = \left( \int_0^\infty f(r\theta)^p r^n \frac{dr}{r} \right)^{1/p} \) Let

\[
K(\phi, \theta) = \int_0^{\theta \cdot \phi} t^{n/p} \frac{dt}{t} = \frac{p'}{n} ((\phi \cdot \theta)^+)^{n/p'}.
\]
where $N^+$ denotes the positive part of the number $N$. Next, we need the following:

**Lemma.** For any $F, G \geq 0$ measurable on $S^{n-1}$ and $K \geq 0$ measurable on $[-1, 1],$

$$\iint_{S^{n-1}^2} F(\theta) G(\phi) K(\theta \cdot \phi) \, d\phi \, d\theta$$

(2.4)

$$\leq \|F\|_{L^p(S^{n-1})} \|G\|_{L^p(S^{n-1})} \int_{S^{n-1}} K(\theta \cdot \phi) \, d\phi.$$

**Proof.** We may assume that all three quantities on the right-hand side of (2.4) are finite. Since $K$ depends only on the inner product $\theta \cdot \phi,$ the integral $\int_{S^{n-1}} K(\theta \cdot \phi) \, d\phi$ is independent of $\theta$. Hölder’s inequality applied to the functions $F$ and 1 with respect to the measure $K(\theta \cdot \phi) \, d\theta$ gives

$$\left(\int_{S^{n-1}} F(\theta) K(\theta \cdot \phi) \, d\theta \right)^{1/p} \left(\int_{S^{n-1}} K(\theta \cdot \phi) \, d\theta \right)^{1/p'}$$

(2.5)

We will now use (2.5) to prove (2.4). The left-hand side of (2.4) is trivially estimated by $\left(\int_{S^{n-1}} F(\theta) K(\theta \cdot \phi) \, d\theta \right)^{1/p} \left(\int_{S^{n-1}} K(\theta \cdot \phi) \, d\theta \right)^{1/p'}$. Applying (2.5) and Fubini’s theorem we bound this last expression by $\|F\|_{L^p(S^{n-1})} \|G\|_{L^p(S^{n-1})} \times \int_{S^{n-1}} K(\theta \cdot \phi) \, d\phi$. The lemma is now proved. Observe that equality is attained in (2.4) if and only if both $F$ and $G$ are constants.

We now continue with the proof of Theorem 2. Applying the lemma and using the fact that $F$ and $G$ have norm one, we estimate (2.3) by $\frac{\omega_n}{\omega_n} \times \int_{S^{n-1}} ((\theta \cdot \phi)^+)^{\frac{p}{p'}} \, d\theta$. To compute this integral, we slice the sphere in the direction transverse to $\phi$. For convenience we may take $\phi = e_1 = (1, 0, \ldots, 0)$. The area of the slice cut by the hyperplane $\phi_1 = s$ is $\omega_{n-2}(1 - s^2)^{\frac{n-2}{2}}$ and the weight of this slice is $(1 - s^2)^{-\frac{1}{2}}$. We get

$$\int_{S^{n-1}} ((\theta \cdot \phi)^+)^{\frac{p}{p'}} \, d\theta = \omega_{n-2} \int_{s=0}^{1} s^{\frac{p}{p'}} (1 - s^2)^{\frac{n-2}{2}} \, ds$$

(2.6)

$$= \omega_{n-2} \frac{1}{2} B\left(\frac{1}{p'}, 1 - \frac{n-2}{2}\right) = \omega_{n-2} \frac{1}{2} B\left(\frac{1}{p'}, 1 - 1\right) = \omega_{n-2} \frac{1}{2} B\left(\frac{n}{p'}, 1\right).$$

We now use that $nv_n = \omega_{n-1}$ to get the final estimate $c_{p,n}$ in (2.2) which completes the proof of (2.1). It remains to establish that the constant $c_{p,n}$ is the best possible. For any $y \in \mathbb{R}^n,$ let $A(y)$ be the spherical cap $\{\theta \in S^{n-1} : |\theta - y| \leq |y|\}$. This cap is nonempty if and only if $|y| \geq 1/2$. For such $y$, the Lebesgue measure $|A(y)|$ is $\omega_{n-2} \int_{1/2|y|}^{1} (1 - s^2)^{\frac{n-2}{2}} \, ds$. Let $G(t) = \chi_{[0,1]}(t) t^{n/p'} \int_{t}^{1} (1 - s^2)^{\frac{n-2}{2}} \, ds$. An easy computation shows that $\|G\|_{L^{1}(\mathbb{R}^n, \nu_n)} = (\frac{2}{n}) \int_{0}^{1} (1 - s^2)^{\frac{n-2}{2}} s^{\frac{n}{2}} \, ds$. Let $h = h_{\epsilon,N}$ be an element of the family.
$|x|^{-n/p} \chi_{|x| \leq N}$ normalized to have $L^p$ norm one. We have

\begin{align*}
\|Sh\|_{L^p(R^n)}^p &= \int_{r=0}^{\infty} \int_{\phi \in S^{n-1}} \left( \frac{1}{v_n r^n} \int_{D(\phi, r)} h(y) dy \right)^p r^{n-1} d\phi dr \\
&= \int_{r=0}^{\infty} \int_{\phi \in S^{n-1}} \left( \frac{1}{v_n r^n} \int_{t=0}^{2r} h(t\theta) t^{n-1} d\theta dt \right)^p r^{n-1} d\phi dr \\
&= \int_{r=0}^{\infty} \int_{\phi \in S^{n-1}} \left( \frac{1}{v_n r^n} \int_{t=0}^{2r} |A((r/t)\phi)| h(t) t^{n-1} \frac{dt}{t} \right)^p r^{n-1} d\phi dr \\
&= \omega_{n-2}^{-p-n} \omega_{n-1} \int_{r=0}^{\infty} \left( \int_{t=0}^{1} h(2rt) (2rt)^{2} G(t) \frac{dt}{t} \right)^p r^{n} dr.
\end{align*}

The convolution inequality $\|g*L\|_{L^p} \leq \|g\|_{L^p} \|L\|_{L^1}$ in the group $(\mathbb{R}^+, \frac{dt}{t})$ written as

\begin{align*}
\int_{r=0}^{\infty} \left( \int_{t=0}^{1} h(2rt) (2rt)^{2} G(t) \frac{dt}{t} \right)^p dr \leq \left( \int_{r=0}^{\infty} h(r) r^{p-n} \frac{dr}{r} \right) \|g\|_{L^p(\mathbb{R}^+)}^p
\end{align*}

becomes an equality as $\epsilon \to 0$ and $N \to \infty$. Inserting (2.8) in (2.7) we obtain

\begin{align*}
\|Sh\|_{L^p(R^n)}^p &\leq \omega_{n-2}^{-p-n} \left( \frac{p'}{n} \right)^p \left( \int_{s=0}^{1} (1 - s^2)^{\frac{n-3}{2}} s^{p-1} ds \right)^p \\
&\times \omega_{n-1} \int_{r=0}^{\infty} h(r) r^{p-n} dr = C_p^p, n
\end{align*}

since $\|h\|_{L^1} = 1$, and equality is attained as $\epsilon \to 0$ and $N \to \infty$. Theorem 2 is now proved.

### 3. A LOWER BOUND FOR THE OPERATOR NORM OF THE HARDY-LITTLEWOOD MAXIMAL FUNCTION ON $L^p(\mathbb{R}^n)$

Let $M(f)(x) = \sup_{r>0} (v_n r^n)^{-1} \int_{|y-x| \leq r} |f(y)| dy$ be the usual Hardy-Littlewood maximal function on $\mathbb{R}^n$. The family of functions $f_{\epsilon,N}(x) = |x|^{-n/p} \chi_{|x| \leq N}$ is extremal for Theorems 1 and 2. Let $A_{p,n}$ be the operator norm of $M$ on $L^p(\mathbb{R}^n)$. By computing $\|M(f_{\epsilon,N})\|_{L^p}/\|f_{\epsilon,N}\|_{L^p}$ and letting $\epsilon \to 0$ and $N \to \infty$ we obtain a lower bound for $A_{p,n}$.

**Proposition.** For $1 < p < \infty$, let $A_{p,n}$ be the best constant $C$ that satisfies the inequality $\|Mf\|_{L^p(R^n)} \leq C\|f\|_{L^p(R^n)}$ for all $f \in L^p$. Then

\begin{align*}
A_{p,n} \geq \frac{p'}{\omega_{n-1}^2} \sup_{\delta > 1} \frac{1}{\delta^n} \int_{-1}^{1} (\sqrt{1 - s^2})^{n-3} (s + \sqrt{s^2 + \delta^2 - 1})^\frac{p}{2} ds
\end{align*}

and the supremum above is attained for some $\delta = \delta_{n,p}$ always less than 2.

**Proof.** The following is only a sketch. Since $|x|^{-n/p}$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$, we can calculate $M(|x|^{-n/p})$ instead. Observe that $M(|x|^{-n/p}) = c |x|^{-n/p}$ where $c = M(|x|^{-n/p})(e_1)$ and $e_1 = (1, 0, \ldots, 0)$. Also note that the supremum of the averages of $|x|^{-n/p}$ over balls of radius $r$ centered at $e_1$ is attained for some $r = 1 + \gamma$ where $\gamma > 0$. We therefore find that

\begin{align*}
c = \sup_{\gamma > 0} \frac{1}{v_n (1 + \gamma)^n} \int_{r=0}^{2+\gamma} r^{n-\frac{p}{2}} A_r \frac{dr}{r},
\end{align*}
where \( A_r = \{ \theta \in S^{n-1} : |r\theta - e_1| < 1 + \gamma \} \). Calculation gives that \( A_r = \omega_{n-1} \) for \( r \leq \gamma \) and \( A_r = \omega_{n-2} \int_{(r^2 - 2y^2)/(2r)}^{1} (1 - s^2)^{2+\frac{1}{\gamma}} ds \) for \( 2 + \gamma > r > \gamma \). We plug these values into (3.2), and we interchange the integration in \( r \) and \( s \):

\[
\int_{r=\gamma}^{2+\gamma} \int_{s=\gamma}^{1} r^{\frac{\gamma}{2}} (1 - s^2)^{\frac{3}{2}+\frac{1}{\gamma}} ds \frac{dr}{r} = \int_{-1}^{1} \int_{r=\gamma}^{r+s+\sqrt{s^2+y^2+2y}} \int_{r+s+\sqrt{s^2+y^2+2y}}^{s} r^{\frac{\gamma}{2}} (1 - s^2)^{\frac{3}{2}+\frac{1}{\gamma}} ds \frac{dr}{r}.
\]

We now let \( \delta = \gamma + 1 \) and obtain (3.1). Note that the constant on the right-hand side of (3.1) reduces to the constant \( c_{p,n} \) of Theorem 2 when \( \delta = 1 \).

4. Final remarks

We end with a couple of remarks. Let \( c_{n,p} \) be the constant of Theorem 2. We observe that \( c_{n,p} \leq \frac{p}{\delta - 1} \). This can be shown directly or via the following inequality which can be found in [HLP]:

\[
\int_{\mathbb{R}^n} f(x)g(x) dx \leq \int_{\mathbb{R}^n} f(x)\tilde{g}(x) dx,
\]

where \( f \) and \( g \) are integrable and \( \tilde{f} \) denotes the symmetric decreasing rearrangement of any function \( f \). Let \( T \) and \( S \) be the operators of Theorems 1 and 2. The nonsymmetric decreasing rearrangement of the kernel of \( S \) is the kernel of \( T \). Taking \( g \) to be the kernel of \( S \) and \( f \) in \( L^p \cap L^1 \) in (4.1), we obtain the pointwise inequality \( Sf \leq Tf \). It follows that \( c_{n,p} \leq \frac{p}{\delta - 1} \).

For any \( \delta > 0 \), we define variants \( T_\delta \) of \( T \) and \( S_\delta \) of \( S \) as follows:

\[
(T_\delta f)(x) = \frac{1}{|D(0, \delta|x|)|} \int_{D(0, \delta|x|)} f(y) dy
\]

and

\[
(S_\delta f)(x) = \frac{1}{|D(x, \delta|x|)|} \int_{D(x, \delta|x|)} f(y) dy.
\]

Since \( (T_\delta f)(x) = (Tf)(\delta x) \), it is immediate that the operator norm of \( T_\delta \) on \( L^p(\mathbb{R}^n) \) is \( \frac{p}{\delta - 1} \delta^{-n/p} \).

To compute the operator norm of \( S_\delta \) on \( L^p(\mathbb{R}^n) \), we repeat the proof of Theorem 2 verbatim. We obtain the following result:

**Theorem.** (A) For \( \delta > 1 \), the operator norm of \( S_\delta \) on \( L^p(\mathbb{R}^n) \) is

\[
p' \frac{\omega_{n-2}}{\omega_{n-1}} \frac{1}{\delta^n} \int_{-1}^{1} (1 - s^2)^{\frac{3}{2}+\frac{1}{\gamma}} (s + \sqrt{s^2 + \delta^2 - 1})^{\frac{\gamma}{2}} ds.
\]

(B) For \( \delta < 1 \), the operator norm of \( S_\delta \) on \( L^p(\mathbb{R}^n) \) is

\[
p' \frac{\omega_{n-2}}{\omega_{n-1}} \frac{1}{\delta^n} \int_{s=\sqrt{1-\delta^2}}^{1} (1 - s^2)^{\frac{3}{2}+\frac{1}{\gamma}} \left[(s + \sqrt{s^2 + \delta^2 - 1})^{\frac{\gamma}{2}} - (s - \sqrt{s^2 + \delta^2 - 1})^{\frac{\gamma}{2}}\right] ds.
\]

(3.1) is of course subsumed in conclusion (A) above.

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