PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

OSCILLATORY SINGULAR INTEGRALS ON HARDY SPACES ASSOCIATED WITH HERZ SPACES

SHANZHEN LU AND DACHUN YANG

(Communicated by J. Marshall Ash)

ABSTRACT. In this paper, it is proved that the oscillatory singular integral operators of nonconvolution type are bounded from Hardy spaces associated with Herz spaces to Herz spaces.

1. INTRODUCTION

Let T be an oscillatory singular integral operator defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy, \tag{1.1}$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and K is a Calderòn-Zygmund kernel.

It is proved by D. H. Phong and E. M. Stein in [6] that T is a bounded operator from H^1_E to L^1 provided $P(x,y)$ is a real bilinear form, where H^1_E is certain variant of the H^1 space. Later, this result is extended into the case of general $P(x,y)$ by Y. B. Pan in [5]. For general $P(x,y)$, it is still an interesting problem whether T is a bounded operator from H^1 to L^1. Recently, some new Hardy spaces HK_p associated with Herz spaces K_p are introduced by the authors in [4] and [8]. The space HK_p is defined by

$$HK_p = \{f : Gf \in K_p\}, \tag{1.2}$$

where Gf is the Grand maximal function of f. An interesting fact shown in [8] is that HK_p is the localization of H^1 at the origin. It is easy to see that the relation between HK_p and K_p is similar to one between H^1 and L^1.

In this paper, we shall prove that T defined by (1.1) is a bounded operator from HK_p to K_p. A counterexample shows that there exists an operator T defined by (1.1), such that T is not a bounded operator from HK_p to itself. To formulate our result, let us first introduce some definitions.

Received by the editors February 16, 1993 and, in revised form, September 5, 1993.
1991 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B30.
Key words and phrases. Oscillatory singular integrals, Herz space, Hardy space.
Research was supported by NSF of China.

Definition 1.1 (see [3]). Let $1 < p < \infty$ and $1/p + 1/p' = 1$. The Herz space $K_p(\mathbb{R}^n)$ consists of those functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n \backslash \{o\})$ for which

$$\|f\|_{K_p} := \sum_{k \in \mathbf{Z}} 2^{kn/p'} \|f\chi_k\|_p < \infty,$$

where $\chi_k = \chi_{C_k}$, $C_k = Q_k \backslash Q_{k-1}$, and $Q_k = \{x : |x| \leq 2^k\}$.

Definition 1.2 (see [4]). Let $1 < p < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(1, p)$-atom if

(1) $\operatorname{Supp} a \subset Q$, where Q is a ball centered at the origin;
(2) $\|a\|_p \leq |Q|^{1/p-1}$;
(3) $\int a(x)\, dx = O$.

Now, we can state our result as follows.

Theorem. *Let $1 < p < \infty$, $P(x, y)$ be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, $\nabla_y P(0, y) = O$, and T be defined as in (1.1). Then T maps $HK_p(\mathbb{R}^n)$ into $K_p(\mathbb{R}^n)$ and*

$$\|Tf\|_{K_p} \leq C\|f\|_{HK_p},$$

where C depends only on the total degree of $P(x, y)$ but not on the coefficients of $P(x, y)$.

2. Proof of the Theorem

To prove the Theorem, we need two lemmas.

Lemma 2.1. *Let $f \in L^1(\mathbb{R}^n)$ and $1 < p < \infty$. Then $f \in HK_p(\mathbb{R}^n)$ if and only if f can be represented as*

$$f(x) = \sum_i \lambda_i a_i(x),$$

where each a_i is a central $(1, p)$-atom and $\sum_i |\lambda_i| < \infty$. Moreover,

$$\|f\|_{HK_p} := \|Gf\|_{K_p} \sim \inf\left\{\sum_i |\lambda_i|\right\},$$

where the infimum is taken over all decompositions of f as above.

See [8] for the proof, and see [4] for other characterizations of HK_p.
The following lemma belongs to Y. B. Pan [5].

Lemma 2.2. *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy*

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 2, \end{cases}$$

and let $\psi \in C_o^\infty(\mathbb{R}^n)$ satisfy

$$\psi(x) = \begin{cases} 1 & \text{for } 1 \leq |x| \leq 2, \\ 0 & \text{for } |x| \leq \frac{1}{4} \text{ or } |x| \geq 4. \end{cases}$$

Define T_k by

$$T_k f(x) = \psi(x/2^k) \int_{\mathbb{R}^n} e^{iP(x,y)} \varphi(y) f(y)\, dy.$$

If $P(x, y)$ satisfies

$$P(x, y) = \sum_{|\alpha|\geq 1,\ |\beta|=l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),$$

where $Q(x, y)$ is a polynomial with degree in y less than or equal to $l-1$, then for each $N > 0$ (large enough) we have

$$\|T_k\|_{L^2\to L^2} \leq C 2^{nk} |a_{\alpha_0\beta_0}|^{-1/2Nl} 2^{-k|\alpha_0|/2Nl},$$

where $|a_{\alpha_0\beta_0}| = \max_{|\alpha|\geq 1,\ |\beta|=l} |a_{\alpha\beta}|$.

Proposition 2.1. *Let $\delta > 0$. Then we have*

$$\|f(\delta\cdot)\|_{K_p} \sim \delta^{-n}\|f\|_{K_p}.$$

Proof. For any $\delta > 0$, there exists a $k_0 \in \mathbb{Z}$ such that $2^{k_0} < \delta \leq 2^{k_0+1}$. By Definition 1.1,

$$\begin{aligned}
\|f(\delta\cdot)\|_{K_p} &= \sum_{k\in\mathbb{Z}} 2^{kn/p'} \left(\int_{C_k} |f(\delta x)|^p\, dx\right)^{1/p} \\
&\leq \sum_{k\in\mathbb{Z}} 2^{kn/p'} \delta^{-n/p} \left(\int_{2^{k+k_0}<|y|\leq 2^{k+k_0+2}} |f(y)|^p\, dy\right)^{1/p} \\
&\leq \delta^{-n/p} 2^{-k_0 n/p'} \sum_{k\in\mathbb{Z}} 2^{(k+k_0)n/p'} \left(\int_{C_{k+k_0}} |f(y)|^p\, dy\right)^{1/p} \\
&\quad + \delta^{-n/p} 2^{-(k_0+1)n/p'} \sum_{k\in\mathbb{Z}} 2^{(k+k_0+1)n/p'} \left(\int_{C_{k+k_0+1}} |f(y)|^p\, dy\right)^{1/p} \\
&\leq C\delta^{-n}\|f\|_{K_p}.
\end{aligned}$$

On the other hand,

$$\|f\|_{K_p} = \|f(\delta^{-1}\delta\cdot)\|_{K_p} \leq C\delta^n \|f(\delta\cdot)\|_{K_p}.$$

This finishes the proof of Proposition 2.1.

By Lemma 2.1, it is easy to see that the proof of the Theorem is reduced to the following proposition.

Proposition 2.2. *Let $1 < p < \infty$, $P(x, y)$ be a real-valued polynomial, $\nabla_y P(0, y) = 0$, and T be defined as in (1, 1). Then for any central $(1, p)$-atom a,*

$$\|Ta\|_{K_p} \leq C,$$

where C is independent of a and the coefficients of $P(x, y)$.

Proof. Let $\operatorname{Supp} a \subset Q$ and Q be a ball centered at the origin with radius δ. If we write $b(x) = \delta^n a(\delta x)$, then $b(x)$ is a central $(1, p)$-atom supporting on unit ball $B(0, 1)$. We also have

$$\begin{aligned}
Ta(\delta x) &= \delta^{-n} T_1 b(x) \\
&:= \delta^{-n}\, \text{p.v.} \int_{\mathbf{R}^n} e^{iP(\delta x, \delta y)} K_1(x-y) b(y)\, dy,
\end{aligned}$$

where $K_1(x) = \delta^n K(\delta x)$. By Proposition 2.1, we obtain

$$\|Ta\|_{K_p} \sim \|T_1 b\|_{K_p}.$$

Let $P_1(x, y) = P(\delta x, \delta y)$. Note that $\nabla_y P_1(0, y) = 0$ and K_1 is also a Calderón-Zygmund kernel. We may assume $T_1 = T$. Thus, it suffices to show

$$\|Tb\|_{K_p} \leq C, \tag{2.1}$$

where C is independent of b and the coefficients of $P(x, y)$ and b is a central $(1, p)$-atom supporting on unit ball $B(0, 1)$.

We now turn to prove (2.1) by using induction on the degree of y, l, in $P(x, y)$. If $l = 0$, then

$$|Tb(x)| = \left|\text{p.v.} \int K(x, y) b(y)\, dy\right|.$$

Thus,

$$\begin{aligned}\|Tb\|_{K_p} &= \sum_{k\in\mathbb{Z}} 2^{kn/p'} \|(Tb)\chi_k\|_p \\ &= \sum_{k\leq 0} \cdots + \sum_{k>0} \cdots := S_1 + S_2.\end{aligned}$$

By L^p-boundedness of Calderón-Zygmund operators,

$$S_1 \leq C \sum_{k\leq 0} 2^{kn/p'} \|b\|_p = C \sum_{k\leq 0} 2^{kn/p'} = C.$$

From the condition of $K(x, y)$,

$$|K(x, y) - K(x, 0)| \leq C|y|/|x - y|^{n+1}, \quad \text{if } |y| < |x - y|/2,$$

it follows that

$$Tb(x) = \int b(y)[K(x, y) - K(x, 0)]\, dy$$

and

$$\begin{aligned}S_2 &= \sum_{k>0} 2^{kn/p'} \|(Tb)\chi_k\|_p \\ &\leq C \sum_{k>0} 2^{kn/p'} \left[\int_{C_k} \left(\int_{B(0,1)} \frac{|b(y)|\,|y|}{|x - y|^{n+1}}\, dy\right)^p dx\right]^{1/p} \\ &\leq C \sum_{k>0} 2^{kn/p'} \left(\int_{C_k} \frac{dx}{|x|^{(n+1)p}}\right)^{1/p} \|b\|_p \\ &\leq C \sum_{k>0} 2^{kn/p'} 2^{-k[(n+1)p-n]/p} = C \sum_{k>0} 2^{-k} = C.\end{aligned}$$

Therefore, (2.1) holds for $l = 0$. Let us now consider the case of $l > 0$. We assume that (2.1) holds for $l - 1$ by induction. Since $\nabla_y P(0, y) = 0$, we can write

$$P(x, y) = \sum_{|\alpha|\geq 1,\ |\beta|=l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),$$

where $Q(x, y)$ is a polynomial with degree in y less than or equal to $l - 1$ and $\nabla_y Q(0, y) = 0$. Denote

$$|a_{\alpha_0\beta_0}| = \max_{|\alpha|\geq 1,\ |\beta|=l} |a_{\alpha\beta}|$$

and

$$r = \max\{3, |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}\}. \tag{2.2}$$

Since $r \geq 3$, we may assume $2^{j_0} < r \leq 2^{j_0+1}$ for some $j_0 \in \mathbb{N}$. We now write

$$\begin{aligned}\|Tb\|_{K_p} &= \sum_{j\leq 0} 2^{jn/p'}\|(Tb)\chi_j\|_p + \sum_{j=1}^{j_0} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &\quad + \sum_{j\geq j_0+1} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

By L^p-boundedness of oscillatory singular integral operators (see [7]), we have

$$I_1 \leq C\sum_{j\leq 0} 2^{jn/p'}\|b\|_p = C\sum_{j\leq 0} 2^{jn/p'} = C.$$

To estimate I_2, we may assume $j_0 \geq 2$. In this case, $r = |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$. By induction assumption,

$$\begin{aligned} I_2 &= \sum_{j=1}^{j_0} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &\leq \sum_{j=1}^{j_0} 2^{jn/p'}\left\{\int_{C_j}\left|\int_{\mathbb{R}^n}(e^{iP(x,y)} - e^{iQ(x,y)})K(x-y)b(y)\,dy\right|^p dx\right\}^{1/p} \\ &\quad + \sum_{j=1}^{j_0} 2^{jn/p'}\left\{\int_{C_j}\left|\int_{\mathbb{R}^n} e^{iQ(x,y)}K(x-y)b(y)\,dy\right|^p dx\right\}^{1/p} \\ &\leq C\sum_{j=1}^{j_0} 2^{jn/p'}\left\{\int_{C_j}\left[\int_{|y|\leq 1}\left|\exp\left(i\sum_{|\alpha|\geq 1,\ |\beta|=l} a_{\alpha\beta}x^\alpha y^\beta\right)\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. -1\right|\frac{|b(y)|}{|x|^n}\,dy\right]^p dx\right\}^{1/p} + C \\ &\leq C\sum_{|\alpha|\geq 1,\ |\beta|=l}|a_{\alpha\beta}|\sum_{j=1}^{j_0} 2^{jn/p'}\left(\int_{C_j}|x|^{(|\alpha|-n)p}\,dx\right)^{1/p} + C \\ &\leq C\sum_{|\alpha|\geq 1,\ |\beta|=l}|a_{\alpha\beta}|\sum_{j=1}^{j_0} 2^{j|\alpha|} + C \\ &\leq C\sum_{|\alpha|\geq 1,\ |\beta|=l}|a_{\alpha\beta}|r^{|\alpha|} + C \\ &\leq C_1|a_{\alpha_0\beta_0}|r^{|\alpha_0|} + C = C_1 + C.\end{aligned}$$

It remains to estimate I_3. Let φ and ψ be the functions as in Lemma 2.2.

Then

$$\begin{aligned} I_3 &= \sum_{j\geq j_0+1} 2^{jn/p'}\|(Tb)\chi_j\|_p \\ &\leq \sum_{j\geq j_0+1} 2^{jn/p'}\left\{\int_{C_j}\left(\int_{\mathbb{R}^n}|K(x-y)-K(x)|\,|b(y)|\,dy\right)^p dx\right\}^{1/p} \\ &\quad + \sum_{j\geq j_0+1} 2^{jn/p'}\left\{\int_{C_j}\frac{1}{|x|^{np}}\left|\int_{\mathbb{R}^n} e^{iP(x,y)}b(y)\,dy\right|^p dx\right\}^{1/p} \\ &\leq \sum_{j\geq j_0+1} 2^{jn/p'}\left(\int_{C_j}\frac{dx}{|x|^{(n+1)p}}\right)^{1/p} + \sum_{j\geq j_0+1} 2^{-jn/p}\|T_jb\|_p \\ &\leq C + \sum_{j\geq j_0+1} 2^{-jn/p}\|T_jb\|_p. \end{aligned}$$

By Lemma 2.2, we have

$$\|T_jb\|_2 \leq C2^{jn/2}|a_{\alpha_0\beta_0}|^{-1/2Nl}2^{-j|\alpha_0|/2Nl}\|b\|_2.$$

It is easy to check from the definition of T_j that

$$\|T_jb\|_\infty \leq C\|b\|_\infty$$

and

$$\|T_jb\|_1 \leq C2^{jn}\|b\|_1.$$

By the interpolation theorem, we obtain

$$\|T_jb\|_p \leq \begin{cases} C2^{jn/p}|a_{\alpha_0\beta_0}|^{-1/Nlp'}2^{-j|\alpha_0|/Nlp'}\|b\|_p & \text{for } 1<p\leq 2, \\ C2^{jn/p}|a_{\alpha_0\beta_0}|^{-1/Nlp}2^{-j|\alpha_0|/Nlp}\|b\|_p & \text{for } 2<p<\infty. \end{cases}$$

It follows from the above and (2.2) that if $1<p\leq 2$, then

$$\begin{aligned} I_3 &\leq C + C|a_{\alpha_0\beta_0}|^{-1/Nlp'}\sum_{j\geq j_0+1} 2^{-j|\alpha_0|/Nlp'} \\ &\leq C + C(|a_{\alpha_0\beta_0}|r^{|\alpha_0|})^{-1/Nlp'} \leq C; \end{aligned}$$

and if $2<p<\infty$, then

$$\begin{aligned} I_3 &\leq C + C|a_{\alpha_0\beta_0}|^{-1/Nlp}\sum_{j\geq j_0+1} 2^{-j|\alpha_0|/Nlp} \\ &\leq C + C(|a_{\alpha_0\beta_0}|r^{|\alpha_0|})^{-1/Nlp} \leq C. \end{aligned}$$

This completes the proof of (2.1) and therefore the proof of Proposition 2.2.

Remark 2.1. Recently, Hardy spaces $HA^p(\mathbb{R}^n)$ related to the Beurling algebras $A^p(\mathbb{R}^n)$ have been introduced by Y. Z. Chen and K. S. Lau in [1] and independently by J. Garcia-Cuerva in [2]. It has been proved by the authors in [4] that

$$HK_p \cap L^p = HA^p$$

and

$$\|f\|_{HA^p} \sim \|f\|_{HK_p} + \|f\|_p. \tag{2.3}$$

On the other hand, it is easy to show that

$$\|f\|_{A^p} \sim \|f\|_{K_p} + \|f\|_p. \tag{2.4}$$

Thus, from (2.3), (2.4), and the Theorem, it is easy to see that under the conditions of Theorem, T defined by (1.1) is a bounded operator from $HA^p(\mathbb{R}^n)$ to $A^p(\mathbb{R}^n)$ and

$$\|Tf\|_{A^p} \leq C\|f\|_{HA^p}.$$

Remark 2.2. A counterexample shows that there exists an operator T defined by (1.1) such that T is not a bounded operator from HK_p to itself. Let us consider $n = 1$. Take a $g \in HK_p(\mathbb{R})$ such that $Hg(x) \neq 0$ a.e., where Hg is the Hilbert transform of g. Let $P(x, y) = tx$, $t \in \mathbb{R}$. Suppose T is a bounded operator from HK_p into itself. Then $Tg \in HK_p(\mathbb{R})$. Thus, by Lemma 2.1, we have

$$\int Tg(x)\,dx = 0.$$

This is

$$\int e^{itx} Hg(x)\,dx = 0, \qquad t \in \mathbb{R}.$$

Hence, $(Hg)^\vee(t) = 0$, $t \in \mathbb{R}$. It has been proved for the case of $l = 0$ in the proof of Theorem that H maps HK_p into K_p. Thus, $Hg \in K_p \subset L^1$. Combining it with $(Hg)^\vee(t) = 0$, $t \in \mathbb{R}$, we get a contradiction,

$$Hg(x) = 0 \quad \text{a.e.}$$

This confirms the above assertion. However, for the oscillatory integral operator T of convolution type with $P(x, y) = P(x - y)$, the second-named author has proved that T maps HK_p into itself provided $\nabla P(0) = 0$. We omit it here.

Acknowledgment

We thank Professor Yibiao Pan for providing us his paper [5].

References

1. Y. Z. Chen and K. S. Lau, *On some new classes of Hardy spaces*, J. Funct. Anal. **84** (1989), 255–278.
2. J. Garcia-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. (2) **39** (1989), 499–513.
3. C. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech. **18** (1968), 283–324.
4. S. Z. Lu and D. C. Yang, *The Littlewood-Paley function and φ-transform characterizations of a new Hardy space HK_2 associated with the Herz space*, Studia Math. **101** (1992), 285–298.
5. Y. B. Pan, *Hardy spaces and oscillatory singular integrals*, Rev. Mat. Iberoamericana **7** (1991), 55–64.
6. D. H. Phong and E. M. Stein, *Hilbert integrals, singular integrals, and Radon transform.* I, Acta Math. **157** (1987), 99–157.
7. F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent group and singular integrals.* I, J. Funct. Anal. **73** (1987), 179–194.
8. D. C. Yang, *The real-variable characterizations of Hardy spaces $HK_p(\mathbb{R}^n)$*, Adv. in Math. (China) (to appear).

Department of Mathematics, Beijing Normal University, Beijing 100875 People's Republic of China