INVARIANT SUBSPACES FOR POSITIVE OPERATORS
ACTING ON A BANACH SPACE WITH BASIS

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(Communicated by Palle E. T. Jorgensen)

Abstract. Recently we established several invariant subspace theorems for operators acting on an $l_p$-space. In this note we extend these results from operators acting on an $l_p$-space to operators acting on any Banach space with a (not necessarily unconditional) Schauder basis. For instance, it is shown that if a continuous quasinilpotent operator on a Banach space is positive with respect to the closed cone generated by a basis, then the operator has a nontrivial closed invariant subspace.

1. Preliminaries

A subset $C$ of a (real or complex) vector space $X$ is said to be a cone whenever $C + C \subseteq C$, $\alpha C \subseteq C$ for each real $\alpha \geq 0$, and $C \cap (-C) = \{0\}$. Every cone $C$ determines a partial order $\leq$ on $X$ by letting $y \leq x$ whenever $x - y \in C$. The notation $x \geq y$ is, of course, equivalent to $y \leq x$. Thus, the cone satisfies $C = \{x \in X : x \geq 0\}$. The elements of $C$ are known as positive vectors. An ordered vector space is a vector space equipped with a cone $C$. For a detailed account about cones and partially ordered vector spaces, we refer the reader to [4].

In this note the word “operator” will be synonymous with “linear operator”. An operator $T: X \to X$ on an ordered vector space is said to be positive (in symbols $T \geq 0$ or $0 \leq T$) if $Tx \geq 0$ for each $x \geq 0$. For a positive operator $T$, it follows that $Ty \leq Tx$ whenever $y \leq x$ holds. For operators, the notation $T \geq S$ means $T - S \geq 0$ or equivalently $Tx \geq Sx$ for each $x \geq 0$.

Recall that a sequence $\{x_n\}$ in a Banach space $X$ is called a Schauder basis (or simply a basis) of $X$ if for every $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. Every basis $\{x_n\}$ gives rise to a natural closed cone $C$ defined by

$$C = \left\{ x = \sum_{n=1}^{\infty} \alpha_n x_n : \alpha_n \geq 0 \text{ for each } n = 1, 2, \ldots \right\}.$$ 

The cone $C$ will be referred to as the closed cone generated by the basis $\{x_n\}$. For an extensive discussion concerning the cone generated by a basis see [5].

Received by the editors July 23, 1993 and, in revised form, September 27, 1993.

1991 Mathematics Subject Classification. Primary 46A40, 46B40, 47B60, 47B65.

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0002-9939/95 $1.00 + .25 per page

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Associated with every basis is the standard sequence of "coefficient functionals". Let \( \{ x_n \} \) be a basis of a Banach space \( X \). Then the linear functional \( f_n \) defined by
\[
 f_n(x) = \alpha_n \quad \text{for each } x = \sum_{i=1}^{\infty} \alpha_i x_i
\]
is a continuous linear functional on \( X \). Observe that each \( f_n \) is also automatically positive with respect to the closed cone generated by the basis \( \{ x_n \} \). Moreover, the sequence of continuous linear functionals \( \{ f_n \} \) satisfies \( f_n(x_m) = \delta_{nm} \).

An operator \( T: X \to X \) on a Banach space with a basis \( \{ x_n \} \) is said to be positive (with respect to this basis) if \( T(C) \subseteq C \), where \( C \) is the closed cone generated by \( \{ x_n \} \). Now fix a basis \( \{ x_n \} \) for a Banach space \( X \). Then every operator \( T: X \to X \) can be identified in the usual manner with an infinite matrix \( [t_{ij}] \). In this context, we can also say that an infinite matrix \( [t_{ij}] \) defines an operator on \( X \). Note that an operator \( T: X \to X \) with matrix \( [t_{ij}] \) is a positive operator if and only if \( t_{ij} \geq 0 \) holds for each pair \( (i, j) \). If the basis \( \{ x_n \} \) is also unconditional, then every positive operator is automatically continuous; see [1, Corollary 2.5, p. 4] or [3, Theorem 12.3, p. 175].

2. INVARIANT SUBSPACES

In this section, we shall extend our invariant subspace results for \( l_p \)-spaces to operators acting on a Banach space with a basis. If a basis is specified, then all notions of positivity will always be with respect to the closed cone generated by this basis. As we shall show, the order structure of a Banach space determined by a basis implies some interesting consequences.

Recall that a continuous operator \( T: X \to X \) on a Banach space is said to be quasi-nilpotent if its spectral radius is zero. It is well known that \( T \) is quasi-nilpotent if and only if \( \lim_{n \to \infty} \| T^n x \|^{1/n} = 0 \) for each \( x \in X \).

**Definition 2.1** ([2]). A continuous operator \( T: X \to X \) on a Banach space is called quasi-nilpotent at a point \( x_0 \) whenever \( \lim_{n \to \infty} \| T^n x_0 \|^{1/n} = 0 \).

A simple example of a one-to-one positive operator on \( l_1 \) that is quasi-nilpotent at a positive vector but is not a quasi-nilpotent operator can be found in [2]. We are now ready to show that on a Banach space with a basis any positive operator that commutes with a positive quasi-nilpotent operator has a nontrivial closed invariant subspace.

**Theorem 2.2.** Let \( X \) be a Banach space with a basis, and let \( T: X \to X \) be a continuous positive operator. If \( T \) commutes with a nonzero positive operator that is quasi-nilpotent at a nonzero positive vector, then \( T \) has a nontrivial closed invariant subspace.

**Proof.** Let \( \{ x_n \} \) be a basis of the Banach space \( X \), and let \( \{ f_n \} \) be the sequence of coefficient functionals associated with the basis \( \{ x_n \} \).

Assume that the nonzero positive operator \( A: X \to X \) satisfies \( TA = AT \) and is quasi-nilpotent at some nonzero positive vector \( y_0 \), i.e., \( \lim_{n \to \infty} \| A^n y_0 \|^{1/n} = 0 \). If \( A y_0 = 0 \), then the kernel of \( A \) is a nontrivial closed subspace that is invariant under \( T \). Thus, we can suppose that \( A y_0 \) is nonzero. By an appropriate scaling of \( y_0 \), we can assume that \( 0 \leq x_k \leq y_0 \) and \( Ax_k \neq 0 \) for some \( k \).
Now let $P : X \to X$ denote the continuous projection onto the vector subspace generated by $x_k$ defined by $P(x) = f_k(x)x_k$. Clearly, $0 \leq Px \leq x$ holds for each $0 \leq x \in X$. We claim that

$$(*) \quad PT^mA x_k = 0$$

for each $m \geq 0$. To see this, fix $m \geq 0$ and let $PT^mA x_k = \alpha x_k$ for some nonnegative scalar $\alpha \geq 0$. Since $P$ is a positive operator and the composition of positive operators is a positive operator, it follows that

$$0 \leq \alpha^n = (PT^mA)^n x_k \leq (T^mA)^n x_k = T^{mn} A^n x_k \leq T^{mn} A^n y_0.$$

Since $f_k$ is a positive linear functional, the above inequality yields

$$0 \leq \alpha^n = f_k(\alpha^n x_k) \leq f_k(T^{mn} A^n y_0).$$

Consequently, $0 \leq \alpha \leq \|f_k\| \|T\|^m \cdot \|A^n y_0\|$, and so

$$0 \leq \alpha \leq \|f_k\|^{1/n} \|T\|^m \cdot \|A^n y_0\|^{1/n}.$$

From $\lim_{n \to \infty} \|A^n y_0\|^{1/n} = 0$, we see that $\alpha = 0$, and thus condition $(*)$ must hold.

Now consider the subspace $Y$ generated by $\{T^mA x_k : m = 0, 1, \ldots\}$. Clearly, $Y$ is invariant under $T$, and since $0 \neq A x_k \in Y$, we see that $Y \neq \{0\}$. Also, for each $y \in Y$, it follows from $(*)$ that

$$f_k(y) = f_k(Py) = 0,$$

and consequently $f_k(y) = 0$ for all $y \in \overline{Y}$. The latter shows that $\overline{Y}$ is a nontrivial closed vector subspace of $X$ that is invariant under the operator $T$, and the proof is complete. $\square$

Corollary 2.3. Let $X$ be a Banach space with a basis. If $T : X \to X$ is a continuous quasinilpotent positive operator, then $T$ has a nontrivial closed invariant subspace.

One can add arbitrary weights to the matrix representing a quasinilpotent positive operator and still be guaranteed that a nontrivial closed invariant subspace exists.

Theorem 2.4. Let $X$ be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on $X$ that is quasinilpotent at a nonzero positive vector. If for a double sequence $\{b_{ij}\}$ of complex numbers the weighted matrix $B = [b_{ij}a_{ij}]$ defines a continuous operator $B$ on $X$, then the operator $B$ has a nontrivial closed invariant subspace.

Proof. Let $\{x_n\}$ be a basis of the Banach space $X$, and let $\{f_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$. Assume that the positive operator $A = [a_{ij}]$ satisfies $\lim_{n \to \infty} \|A^n y_0\|^{1/n} = 0$ for some nonzero positive vector $y_0 \neq 0$. An appropriate scaling of $y_0$ shows that there exists some $k$ satisfying $0 \leq x_k \leq y_0$. If $A x_k = 0$, then an easy argument shows that $B x_k = 0$, and thus the kernel of $B$ is a nontrivial closed invariant subspace (here we assume, of course, that $B \neq 0$). Thus, we can suppose that $A x_k$ is nonzero.

Now let $P : X \to X$ denote the positive projection defined by $P(x) = f_k(x)x_k$. Then arguing as in the proof of Theorem 2.2, we can establish that
$PA^mx_k = 0$ for each $m \geq 1$. In particular, we have $f_k(A^mx_k) = 0$ for each $m \geq 1$. Consequently, for each $m \geq 1$ and for each positive operator $S: X \to X$ satisfying $0 \leq S \leq A^m$ we have

$$0 \leq f_k(Sx_k) \leq f_k(A^mx_k) = 0.$$

Next, consider the vector subspace $Y$ generated by the set

$$\{Sx_k: \exists S \text{ such that } 0 < S \leq A^m \text{ for some } m \geq 1\}.$$

Clearly, $Y$ is invariant under each operator $R$ that satisfies $0 \leq R \leq A$. Also, from (**) it follows that

$$f_k(y) = 0$$

for all $y \in \overline{Y}$. The latter shows that $\overline{Y}$ is a nontrivial closed vector subspace of $X$ that is invariant under each operator $R: X \to X$ satisfying $0 \leq R \leq A$.

Next, consider the operator $A_{ij}$ defined by $A_{ij}(x_j) = a_{ij}x_j$ and $A_{ij}(x_m) = 0$ for $m \neq j$. Since the operator satisfies $0 \leq A_{ij} \leq A$, it follows that $\overline{Y}$ is invariant under each of the operator $A_{ij}$. Therefore, the vector subspace $\overline{Y}$ is invariant under the operators

$$B_n = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}A_{ij}.$$

However, the sequence of operators $\{B_n\}$ converges in the strong operator topology to $B$. Therefore, $B(\overline{Y}) \subset \overline{Y}$ holds, and thus the operator $B$ has a nontrivial closed invariant subspace. □

**Corollary 2.5.** Let $X$ be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on $X$ which is quasinilpotent at a nonzero positive vector. If a continuous operator $T: X \to X$ is defined by a matrix $T = [t_{ij}]$ satisfying $t_{ij} = 0$ whenever $a_{ij} = 0$, then the operator $T$ has a nontrivial closed invariant subspace.

We conclude with two remarks.

(1) Consider a quasinilpotent operator on a Banach space with a basis. Suppose the operator is not positive with respect to this basis. At first glance, it appears that our invariant subspace theorems do not apply. However, if one considers a change of basis, then the operator might become positive with respect to the new basis, and therefore, it would have a nontrivial closed invariant subspace. It would be interesting to find out when a given quasinilpotent operator on a Hilbert space can be made positive with respect to some basis.

(2) It is well known that if a Banach space $X$ has an unconditional basis, then (up to an equivalent norm) $X$ is a discrete Banach lattice. Therefore, some of the results obtained in [2] are indeed special cases of the results obtained here.

**References**


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