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INVARIANT SUBSPACES FOR POSITIVE OPERATORS ACTING ON A BANACH SPACE WITH BASIS

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ABSTRACT. Recently we established several invariant subspace theorems for operators acting on an l_p-space. In this note we extend these results from operators acting on an l_p-space to operators acting on any Banach space with a (not necessarily unconditional) Schauder basis. For instance, it is shown that if a continuous quasinilpotent operator on a Banach space is positive with respect to the closed cone generated by a basis, then the operator has a nontrivial closed invariant subspace.

1. PRELIMINARIES

A subset C of a (real or complex) vector space X is said to be a cone whenever $C + C \subseteq C$, $\alpha C \subseteq C$ for each real $\alpha \geq 0$, and $C \cap (-C) = \{0\}$. Every cone C determines a partial order $\leq$ on X by letting $y \leq x$ whenever $x - y \in C$. The notation $x \geq y$ is, of course, equivalent to $y \leq x$. Thus, the cone satisfies $C = \{x \in X : x \geq 0\}$. The elements of C are known as positive vectors. An ordered vector space is a vector space equipped with a cone C. For a detailed account about cones and partially ordered vector spaces, we refer the reader to [4].

In this note the word "operator" will be synonymous with "linear operator". An operator $T: X \to X$ on an ordered vector space is said to be positive (in symbols $T \geq 0$ or $0 \leq T$) if $Tx \geq 0$ for each $x \geq 0$. For a positive operator T, it follows that $Ty \leq Tx$ whenever $y \leq x$ holds. For operators, the notation $T \geq S$ means $T - S \geq 0$ or equivalently $Tx \geq Sx$ for each $x \geq 0$.

Recall that a sequence $\{x_n\}$ in a Banach space X is called a Schauder basis (or simply a basis) of X if for every $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. Every basis $\{x_n\}$ gives rise to a natural closed cone C defined by

$$C = \left\{ x = \sum_{n=1}^{\infty} \alpha_n x_n : \alpha_n \geq 0 \text{ for each } n = 1, 2, \ldots \right\}.$$

The cone C will be referred to as the closed cone generated by the basis $\{x_n\}$. For an extensive discussion concerning the cone generated by a basis see [5].

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Associated with every basis is the standard sequence of "coefficient functionals". Let $\{x_n\}$ be a basis of a Banach space X. Then the linear functional f_n defined by

$$f_n(x) = \alpha_n \quad \text{for each } x = \sum_{i=1}^{\infty} \alpha_i x_i$$

is a continuous linear functional on X. Observe that each f_n is also automatically positive with respect to the closed cone generated by the basis $\{x_n\}$. Moreover, the sequence of continuous linear functionals $\{f_n\}$ satisfies $f_n(x_m) = \delta_{nm}$.

An operator $T: X \to X$ on a Banach space with a basis $\{x_n\}$ is said to be positive (with respect to this basis) if $T(C) \subseteq C$, where C is the closed cone generated by $\{x_n\}$. Now fix a basis $\{x_n\}$ for a Banach space X. Then every operator $T: X \to X$ can be identified in the usual manner with an infinite matrix $[t_{ij}]$. In this context, we can also say that an infinite matrix $[t_{ij}]$ defines an operator on X. Note that an operator $T: X \to X$ with matrix $[t_{ij}]$ is a positive operator if and only if $t_{ij} \geq 0$ holds for each pair (i, j). If the basis $\{x_n\}$ is also unconditional, then every positive operator is automatically continuous; see [1, Corollary 2.5, p. 4] or [3, Theorem 12.3, p. 175].

2. Invariant subspaces

In this section, we shall extend our invariant subspace results for l_p-spaces to operators acting on a Banach space with a basis. If a basis is specified, then all notions of positivity will always be with respect to the closed cone generated by this basis. As we shall show, the order structure of a Banach space determined by a basis implies some interesting consequences.

Recall that a continuous operator $T: X \to X$ on a Banach space is said to be *quasinilpotent* if its spectral radius is zero. It is well known that T is quasinilpotent if and only if $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$ for each $x \in X$.

Definition 2.1 ([2]). A continuous operator $T: X \to X$ on a Banach space is called *quasinilpotent at a point* x_0 whenever $\lim_{n\to\infty} \|T^n x_0\|^{1/n} = 0$.

A simple example of a one-to-one positive operator on l_1 that is quasinilpotent at a positive vector but is not a quasinilpotent operator can be found in [2]. We are now ready to show that on a Banach space with a basis any positive operator that commutes with a positive quasinilpotent operator has a nontrivial closed invariant subspace.

Theorem 2.2. *Let X be a Banach space with a basis, and let $T: X \to X$ be a continuous positive operator. If T commutes with a nonzero positive operator that is quasinilpotent at a nonzero positive vector, then T has a nontrivial closed invariant subspace.*

Proof. Let $\{x_n\}$ be a basis of the Banach space X, and let $\{f_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$.

Assume that the nonzero positive operator $A: X \to X$ satisfies $TA = AT$ and is quasinilpotent at some nonzero positive vector y_0, i.e., $\lim_{n\to\infty} \|A^n y_0\|^{1/n} = 0$. If $Ay_0 = 0$, then the kernel of A is a nontrivial closed subspace that is invariant under T. Thus, we can suppose that Ay_0 is nonzero. By an appropriate scaling of y_0, we can assume that $0 \leq x_k \leq y_0$ and $Ax_k \neq 0$ for some k.

Now let $P\colon X \to X$ denote the continuous projection onto the vector subspace generated by x_k defined by $P(x) = f_k(x)x_k$. Clearly, $0 \le Px \le x$ holds for each $0 \le x \in X$. We claim that

$$(*) \qquad PT^m A x_k = 0$$

for each $m \ge 0$. To see this, fix $m \ge 0$ and let $PT^m A x_k = \alpha x_k$ for some nonnegative scalar $\alpha \ge 0$. Since P is a positive operator and the composition of positive operators is a positive operator, it follows that

$$0 \le \alpha^n x_k = (PT^m A)^n x_k \le (T^m A)^n x_k = T^{mn} A^n x_k \le T^{mn} A^n y_0 .$$

Since f_k is a positive linear functional, the above inequality yields

$$0 \le \alpha^n = f_k(\alpha^n x_k) \le f_k(T^{mn} A^n y_0) .$$

Consequently, $0 \le \alpha^n \le \|f_k\| \, \|T\|^{mn} \cdot \|A^n y_0\|$, and so

$$0 \le \alpha \le \|f_k\|^{1/n} \|T\|^m \cdot \|A^n y_0\|^{1/n} .$$

From $\lim_{n\to\infty} \|A^n y_0\|^{1/n} = 0$, we see that $\alpha = 0$, and thus condition $(*)$ must hold.

Now consider the subspace Y generated by $\{T^m A x_k \colon m = 0, 1, \ldots\}$. Clearly, Y is invariant under T, and since $0 \ne Ax_k \in Y$, we see that $Y \ne \{0\}$. Also, for each $y \in Y$, it follows from $(*)$ that

$$f_k(y) = f_k(Py) = 0 ,$$

and consequently $f_k(y) = 0$ for all $y \in \overline{Y}$. The latter shows that $\overline{Y}$ is a nontrivial closed vector subspace of X that is invariant under the operator T, and the proof is complete. □

Corollary 2.3. *Let X be a Banach space with a basis. If $T\colon X \to X$ is a continuous quasinilpotent positive operator, then T has a nontrivial closed invariant subspace.*

One can add arbitrary weights to the matrix representing a quasinilpotent positive operator and still be guaranteed that a nontrivial closed invariant subspace exists.

Theorem 2.4. *Let X be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on X that is quasinilpotent at a nonzero positive vector. If for a double sequence $\{b_{ij}\}$ of complex numbers the weighted matrix $B = [b_{ij}a_{ij}]$ defines a continuous operator B on X, then the operator B has a nontrivial closed invariant subspace.*

Proof. Let $\{x_n\}$ be a basis of the Banach space X, and let $\{f_n\}$ be the sequence of coefficient functionals associated with the basis $\{x_n\}$. Assume that the positive operator $A = [a_{ij}]$ satisfies $\lim_{n\to\infty} \|A^n y_0\|^{1/n} = 0$ for some nonzero positive vector $y_0 \ne 0$. An appropriate scaling of y_0 shows that there exists some k satisfying $0 \le x_k \le y_0$. If $Ax_k = 0$, then an easy argument shows that $Bx_k = 0$, and thus the kernel of B is a nontrivial closed invariant subspace (here we assume, of course, that $B \ne 0$). Thus, we can suppose that Ax_k is nonzero.

Now let $P\colon X \to X$ denote the positive projection defined by $P(x) = f_k(x)x_k$. Then arguing as in the proof of Theorem 2.2, we can establish that

$PA^m x_k = 0$ for each $m \geq 1$. In particular, we have $f_k(A^m x_k) = 0$ for each $m \geq 1$. Consequently, for each $m \geq 1$ and for each positive operator $S: X \to X$ satisfying $0 \leq S \leq A^m$ we have

$$(**) \qquad 0 \leq f_k(Sx_k) \leq f_k(A^m x_k) = 0.$$

Next, consider the vector subspace Y generated by the set

$$\{Sx_k : \exists S \text{ such that } 0 \leq S \leq A^m \text{ for some } m \geq 1\}.$$

Clearly, Y is invariant under each operator R that satisfies $0 \leq R \leq A$. Also, from $(**)$, it follows that

$$f_k(y) = 0$$

for all $y \in \overline{Y}$. The latter shows that $\overline{Y}$ is a nontrivial closed vector subspace of X that is invariant under each operator $R: X \to X$ satisfying $0 \leq R \leq A$.

Next, consider the operator A_{ij} defined by $A_{ij}(x_j) = a_{ij}x_j$ and $A_{ij}(x_m) = 0$ for $m \neq j$. Since the operator satisfies $0 \leq A_{ij} \leq A$, it follows that $\overline{Y}$ is invariant under each of the operator A_{ij}. Therefore, the vector subspace $\overline{Y}$ is invariant under the operators

$$B_n = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} A_{ij}.$$

However, the sequence of operators $\{B_n\}$ converges in the strong operator topology to B. Therefore, $B(\overline{Y}) \subset \overline{Y}$ holds, and thus the operator B has a nontrivial closed invariant subspace. □

Corollary 2.5. *Let X be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on X which is quasinilpotent at a nonzero positive vector. If a continuous operator $T: X \to X$ is defined by a matrix $T = [t_{ij}]$ satisfying $t_{ij} = 0$ whenever $a_{ij} = 0$, then the operator T has a nontrivial closed invariant subspace.*

We conclude with two remarks.

(1) Consider a quasinilpotent operator on a Banach space with a basis. Suppose the operator is not positive with respect to this basis. At first glance, it appears that our invariant subspace theorems do not apply. However, if one considers a change of basis, then the operator might become positive with respect to the new basis, and therefore, it would have a nontrivial closed invariant subspace. It would be interesting to find out when a given quasinilpotent operator on a Hilbert space can be made positive with respect to some basis.

(2) It is well known that if a Banach space X has an unconditional basis, then (up to an equivalent norm) X is a discrete Banach lattice. Therefore, some of the results obtained in [2] are indeed special cases of the results obtained here.

References

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