ANTIPODAL COINCIDENCE
FOR MAPS OF SPHERES INTO COMPLEXES

MAREK IZYDOREK AND JAN JAWOROWSKI

(Communicated by Thomas Goodwillie)

Abstract. This paper gives a partial answer to the question of whether there exists a Borsuk-Ulam type theorem for maps of $S^n$ into lower-dimensional spaces, which are not necessarily manifolds. It is shown that for each $k$ and $n < 2k - 1$, there exists a map $f$ of $S^n$ into a contractible $k$-dimensional complex $Y$ such that $fx \neq f(-x)$, for all $x \in S^n$. In particular, there exists a map of $S^3$ into a 2-dimensional complex $Y$ without an antipodal coincidence. This answers a question raised by Conner and Floyd in 1964. The complex $Y$ provides also an example of a contractible $k$-dimensional complex whose deleted product has the Yang-index equal to $2k - 1$.

1. Introduction and notation

We will say that a map $f$ from $S^n$ to a space $Y$ has an antipodal coincidence if there exists a point $x \in S^n$ such that $fx = f(-x)$. The classical Borsuk-Ulam theorem says that every map $f: S^n \to \mathbb{R}^k$ has an antipodal coincidence if $k \leq n$. Conner and Floyd ([1], p. 85) proved that if $k < n$, then every map $f: S^n \to M^k$ of $S^n$ into a $k$-dimensional differentiable manifold $M^k$ has an antipodal coincidence. They also asked a specific question ([1], 89) of whether there is a map of $S^3$ into a 2-dimensional complex without an antipodal coincidence. We will construct such a map of $S^3$ into a 2-dimensional contractible complex $Y$. The construction is extremely simple and geometric: in our example, $Y$ is a subcomplex of the barycentric subdivision of a standard simplex. We will also show that 3 is the lowest dimension of the sphere for which such an example can be constructed.

If $A$ and $B$ are spaces, we will denote by $A \ast B$ the join of $A$ and $B$. It consists of segments joining the points of $A$ with the points of $B$, the segments being mutually disjoint except at the endpoints. The join of $A$ with the empty set is $A$ itself. There is a standard (deformation) retraction of $(A \ast B) - A$ to $B$ along the segments of the join.

If $s$ is a simplex and $t$ is a face of $s$, we will denote by $c(t)$ the face of $s$ which is complementary to $t$; i.e., $s$ is the join of $t$ and $c(t)$; and

Received by the editors June 8, 1993 and, in revised form, September 28, 1993 and October 12, 1993.

1991 Mathematics Subject Classification. Primary 55M20; Secondary 55M35.

Work by the first author was supported, in part, by the Kosciuszko Foundation.

©1995 American Mathematical Society
0002-9939/95 $1.00 + .25$ per page

1947
\[ \dim c(t) = \dim s - \dim t - 1. \] The barycenter of a simplex \( s \) will be denoted by \( b(s) \). By the \textit{carrier} of a point \( x \) in a complex \( K \) we mean, as usual, the simplex of \( K \) containing \( x \) in the interior.

Let \( \Delta \) be a standard \((n + 1)\)-dimensional simplex and let \( \Delta' \) denote its barycentric subdivision. We can think of \( \Delta \) as a subset of \( \mathbb{R}^{n+2} \) embedded in the standard way, so that the coordinates in \( \mathbb{R}^{n+2} \) are the barycentric coordinates of points of \( \Delta \). We will denote by \( \alpha : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} \) the symmetry of \( \mathbb{R}^{n+2} \) about the diagonal of \( \mathbb{R}^{n+2} \), i.e., the line where all the coordinates are equal to each other. Then \( \alpha \) corresponds to the antipodal map of the \((n + 1)\)-plane \( P \) in which the simplex \( \Delta \) lies. Let \( o \) be the barycenter of \( \Delta \). Two points of \( \mathbb{R}^{n+2} \) are said to be \textit{antipodal} (to each other) if they lie on opposite sides of the center \( o \). The simplex \( \Delta \) is not invariant under the central symmetry \( \alpha \), but the symmetry induces a simplicial map \( \beta : \Delta' \to \Delta' \) of the barycentric subdivision of \( \Delta \). In fact, if \( b = b(s) \) is a vertex of \( \Delta' \) and \( b \) is the barycenter of a face \( s \) of \( \Delta \), then \( \beta(b) \) is the barycenter of the face complementary to \( b : \beta(b) = b(c(s)) \).

The vertices of the barycentric subdivision \( \Delta' \) are partially ordered, as usual, by their “ranks”: \textit{the rank} of a vertex \( b \) of \( \Delta' \) is the dimension of the face of \( \Delta \) of which \( b \) is the barycenter. Note the following facts:

\textbf{Lemma 1.} (1) The involution \( \beta : \Delta' \to \Delta' \) reverses the ranks of the vertices: if \( b \) is a vertex of \( \Delta' \) of rank \( q \), then the rank of \( \beta(b) \) is \( n - q \) (in the case \( q = n + 1 \) it is understood that the center \( o \) is the barycenter of both \( \Delta \) and the empty simplex).

(2) \( \beta : \Delta' \to \Delta' \) is simplicially free outside the center \( o \) in the sense that for each simplex \( s \) of \( \Delta' \), the intersection of \( s \) with \( \beta(s) \) consists at most of the center \( o \).

(3) If \( x \) and \( y \) is an antipodal pair in \( \Delta \) and if \( u \) and \( v \) are the carriers of \( x \) and \( y \) in \( \Delta' \), respectively, then \( v = \beta(u) \). \( \Box \)

The construction of our example rests on the fact that a simplex is so much “antisymmetric” with respect to the antipodal map.

\textbf{Lemma 2.} If \( s \) is a \( q \)-simplex of \( \Delta' \) which is a part of the \( q \)-skeleton \( \Delta^q \) of \( \Delta \), then the intersection \( (\beta(s)) \cap \Delta^q \) is a face of \( \beta(s) \) of dimension \( 2q - n \).

\textbf{Proof.} Since \( s \) is a simplex of \( \Delta' \) in \( \Delta^q \), the ranks of its vertices form a sequence \( (0, \ldots, q) \). Thus by Lemma 1, the ranks of \( \beta(s) \) form a sequence \( (n - q, \ldots, n) \). The face of \( \beta(s) \) which is in \( \Delta^q \) is made up of the vertices whose ranks are not more than \( q \). In the sequence \( (n - q, \ldots, n) \) there are exactly \( 2q - n + 1 \) integers not greater than \( q \). \( \Box \)

\textbf{Corollary.} If \( 2q + 1 \leq n \), then \( \beta(\Delta^q) \cap \Delta^q = \emptyset \).

2. Construction of the complex \( Y \)

Throughout Sections 2 and 3, let \( q \) be an integer and \( k = n - q \).

\textbf{Definition.} Every simplex \( s \) of the barycentric subdivision \( \Delta' \) is the join of a unique pair \( u, v \) of simplices of \( \Delta' \), where \( u \) lies in the \( q \)-skeleton \( \Delta^q \) of \( \Delta \) and \( v \) is disjoint from the \( q \)-skeleton. The simplices \( v \) so obtained form a \( k \)-dimensional subcomplex \( Y^k \) of \( \Delta' \). We will call it the \( k \)-\textit{dimensional spine}, denoted \( Y^k \), of \( \Delta \) (briefly, the \( k \)-\textit{spine}) “dual” to the \( q \)-skeleton \( \Delta^q \). We can
also define the spine inductively by saying that the $k$-spine of a simplex is the union of the cones over the $(k-1)$-spines of the boundary simplices.

Let $v_0, \ldots, v_{n+1}$ be the vertices of $\Delta$. Given a set of $q+1$ indices $i_0, \ldots, i_q$ out of $(0, \ldots, n+1)$, we will denote by $b(i_0, \ldots, i_q)$ the center of the face spanned by $v_{i_0}, \ldots, v_{i_q}$. The $k$-spine $Y^k$ is disjoint with the $q$-skeleton of $\Delta$ and intersects each $(q+1)$-face of $\Delta$ spanned by $v_{i_0}, \ldots, v_{i_{q+1}}$ in its center $b(i_0, \ldots, i_{q+1})$. These centers are "the extremities" of the spine $Y^k$. There are $(n+2)^{q+2}$ of them.

Let $C_k$ be the convex hull of the $k$-spine $Y^k$. Then $C_k$ is also the convex hull of the set of its extremities \{b(i_0, \ldots, i_{q+1})\}. It is a convex $(n+1)$-dimensional polyhedron with vertices \{b(i_0, \ldots, i_{q+1})\} contained in $\Delta - \Delta^q$. The boundary of $C_k$, which we denote by $S_k$, is a simplicial $n$-sphere.

Note that $C_k$ is the intersection of the $(n+2)$ half-spaces given by $(q+2)x_i \leq 1$, $i = 0, \ldots, n+1$, with the simplex $\Delta$. The part of $C_k$ lying on an $m$-face of $C_k$ is just $C_{k+m-n-1}$.

The special case of $n = 3, k = 2, q = 1$. In this case (which is actually the first interesting one), we can see that $Y^2$ is a 2-dimensional complex whose intersection with each 3-dimensional simplex $s$ on the boundary of $\Delta$ consists of four intervals going from the center of $s$ to its 2-dimensional faces. The complex $Y^2$ itself is the cone over the graph made up by these four-legged objects. The convex hull $C_2$ of $Y^2$ is a 4-dimensional polyhedron with 10 vertices. It cannot be a regular polyhedron since there are no regular 4-dimensional polyhedra with 10 vertices. In fact, the boundary $S_2$ of $C_2$ has five tetrahedra (one in every 3-simplex of $S_2$), five octahedra (one "opposite" each vertex of $\Delta$), thirty triangles, thirty edges and ten vertices (the Euler number being zero, as it should be).

3. THE MAP OF $S^n$ INTO $Y^k$

Recall that every simplex $s$ of $\Delta'$ is the join of a unique simplex $u$ of the barycentric subdivision of the $q$-skeleton of $\Delta'$ of $\Delta$ with a unique simplex $v$ of $Y^k$. Thus there exists a standard deformation retraction of $\Delta - \Delta^q$ to $Y^k$ along the lines of the join in each simplex of $\Delta'$ (with its face in $\Delta^q$ deleted). Let $f: \Sigma_k \cong S^n \to Y^k$ be the restriction of this map to $\Sigma_k$. The antipodal symmetry $\alpha$ induces a free involution on $\Sigma_k \cong S^n$ which will also be called antipodal.

Lemma 3. If $x$ and $y$ is a pair of antipodal points of $\Sigma_k$ such that $fx = fy$, then $fx = fy$ must be the center $o$.

Proof. Let $z = fx = fy$. Let $t$ be the carrier of $z$ in $Y^k$ and let $u$ and $v$ be the carriers of $x$ and $y$, respectively, in $\Delta'$. By the construction of $f$, the simplex $t$ is a face of both $u$ and $v$. Since $x$ and $y$ is an antipodal pair, Lemma 1 implies that the intersection $u \cap v$ is just the center $o$. Thus $z = o$. □

Proposition. If $n < 2k - 1$, then the map $f: S^n \to Y^k$ has no antipodal coincidence: if $x$ and $y$ is an antipodal pair on $S^n$, then $fx \neq fy$.

Proof. By Lemma 3, $f$ projects the points $x$ and $y$ to the center $o$ from an antipodal pair of points lying in the $(n-k)$-skeleton $\Delta^{n-k}$ of $\Delta$. However, if
4. Concluding remarks

It is not hard to see that if \( n > 2k \), then every map \( f: S^n \to Y \) of a finite \( k \)-dimensional complex \( Y \) into \( S^n \) has an antipodal coincidence. If, in addition, \( H_k(Y; \mathbb{Z}_2) = 0 \), then the inequality \( n > 2k - 1 \) is already sufficient. For, if \( f: S^n \to Y \) is a map without an antipodal coincidence, then \( f \) defines a \( \mathbb{Z}_2 \)-equivariant map from \( S^n \) to the deleted product \( Y^* \) of \( Y \). By an argument similar to that used in [2] one can show that \( H^{2k}(Y^*; \mathbb{Z}_2) = 0 \). Thus the Yang \( \mathbb{Z}_2 \)-index, \( \text{Ind}^{\mathbb{Z}_2} Y^* \), of \( Y^* \) (see [1] and [3]) is less than \( 2k < n \) and hence an equivariant map from \( S^n \) to \( Y^* \) cannot exist.

Thus we have the following result.

**Theorem.** For each \( k \) and \( n \leq 2k - 1 \), there exists a map \( f \) of \( S^n \) into a contractible \( k \)-dimensional complex \( Y \) without an antipodal coincidence. In particular, there exists such a map of \( S^3 \) into a contractible 2-dimensional complex. Moreover, \( n = 3 \) is the lowest integer for which there is a map of \( S^n \) into a complex of dimension less than \( n \) without an antipodal coincidence. \( \square \)

In particular, the deleted product \( Y^* \) of the contractible complex \( Y \) constructed in Section 2 has the Yang \( \mathbb{Z}_2 \)-index equal to \( 2k - 1 \). We do not know whether there exist a \( k \)-dimensional complex whose deleted product has the Yang \( \mathbb{Z}_2 \)-index equal to \( 2k \).

**References**


WYDZIAL FIZYKI TECHNICZNEJ I MATEMATYKI STOSOWANEJ, POLITECHNIKA GDANSKA, UL. MAJAKOWSKIEGO 11-12, 80-952 GDANSK, POLAND

E-mail address: izydorek@iu-math.math.indiana.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405-5701

Current address: Department of Mathematics, The University of Auckland, Auckland, New Zealand

E-mail address: jaworows@uics.indiana.edu