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NORMAL DERIVATIONS IN NORM IDEALS

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ABSTRACT. We establish the orthogonality of the range and the kernel of a normal derivation with respect to the unitarily invariant norms associated with norm ideals of operators. Related orthogonality results for certain nonnormal derivations are also given.

1. INTRODUCTION

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex separable Hilbert space H. For operators A, B in $B(H)$, the generalized derivation $\delta_{A,B}$ as an operator on $B(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB \quad \text{for all } X \in B(H). \tag{1}$$

When $A = B$, we simply write δ_A for $\delta_{A,A}$. If N is a normal operator in $B(H)$, then δ_N is said to be a normal derivation.

In his investigation of normal derivations, Anderson [1, Theorem 1.7] proved that if N and S are operators in $B(H)$ such that N is normal and $NS = SN$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\| \geq \|S\|, \tag{2}$$

where $\|\cdot\|$ is the usual operator norm. Thus in the sense of [1, Definition 1.2], inequality (2) says that the range of δ_N is orthogonal to the kernel of δ_N, which is just the commutant $\{N\}'$ of N.

It has been shown in [11, Theorem 1] that if N and S are operators in $B(H)$ such that N is normal, S is a Hilbert-Schmidt operator, and $S \in \{N\}'$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2, \tag{3}$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. Thus in the usual Hilbert space sense, the Hilbert-Schmidt operators in the range of δ_N are orthogonal to those in the kernel of δ_N.

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It has also been shown recently in [12, Theorem 3.2] that if N and S are operators in $B(H)$ such that N is normal and S belongs to some Schatten p-class C_p with $1 \leq p \leq \infty$ and $S \in \{N\}'$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\|_p \geq \|S\|_p . \tag{4}$$

The usual operator norm, the Hilbert-Schmidt norm, and the Schatten p-norms are only examples of a large family of unitarily invariant (or symmetric) norms on $B(H)$.

The purpose of this paper is to investigate the orthogonality of the range and the kernel of a normal derivation with respect to the wider class of unitarily invariant norms on $B(H)$. Derivations induced by certain nonnormal operators will also be discussed.

In §2 we will use a completely different analysis to extend (4) to all unitarily invariant norms defined on norm ideals of compact operators in $B(H)$. Extensions of this result to certain nonnormal operators will be the main theme of §3, in which we will treat derivations of the form $\delta_{A,B}$, where A is a dominant operator and B^* is M-hyponormal. Moreover we will discuss the validity of (2) for various classes of derivations at the expense of requiring that S is normal. A relevant example will also be presented.

Recall that each unitarily invariant norm $|||\cdot|||$ is defined on a natural subclass $J_{|||\cdot|||}$ of $B(H)$ called the norm ideal associated with the norm $|||\cdot|||$ and satisfies the invariance property $|||UAV||| = |||A|||$ for all $A \in J_{|||\cdot|||}$ and for all unitary operators $U, V \in B(H)$. While the usual operator norm $\|\cdot\|$ is defined on all of $B(H)$, the other unitarily invariant norms are defined on norm ideals contained in the ideal of compact operators in $B(H)$. Given any compact operator $A \in B(H)$, denote by $s_1(A) \geq s_2(A) \geq \cdots$ the singular values of A, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$. There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if $|||\cdot|||$ is a unitarily invariant norm, then there is a unique symmetric gauge function Φ such that

$$|||A||| = \Phi(\{s_j(A)\}) \tag{5}$$

for all $A \in J_{|||\cdot|||}$.

For $1 \leq p \leq \infty$, define

$$\|A\|_p = \left(\sum_j s_j^p(A)\right)^{1/p}, \tag{6}$$

where, by convention, $\|A\|_\infty = s_1(A)$ is the usual operator norm of the compact operator A. These unitarily invariant norms are the well-known Schatten p-norms associated with the Schatten p-classes C_p, $1 \leq p \leq \infty$. Hence C_1, C_2, and C_∞ are the trace class, the Hilbert-Schmidt class, and the class of compact operators, respectively. For good accounts on the theory of norm ideals and their associated unitarily invariant norms, the reader is referred to [9], [13], or [14] (see also [2] and references therein).

2. Normal derivations

In this section we present our main result of this paper. This result asserts that if N is a normal operator in $B(H)$, then with respect to any unitarily invariant norm $|||\cdot|||$, $\operatorname{ran}\delta_N \cap J_{|||\cdot|||}$ is orthogonal to $\ker\delta_N \cap J_{|||\cdot|||}$, where $\operatorname{ran}\delta_N$ and $\ker\delta_N$ are the range and the kernel of δ_N, respectively.

To accomplish our goal we need two lemmas.

Lemma 1. *Let $N \in B(H)$ be diagonal (normal with pure point spectrum), $S \in \{N\}'$, and $X \in B(H)$. If $\delta_N(X) + S \in J_{|||\cdot|||}$, then $S \in J_{|||\cdot|||}$ and*

$$|||\delta_N(X) + S||| \geq |||S|||. \tag{7}$$

Proof. Let N have the distinct eigenvalues $\lambda_1, \lambda_2, \dots$. Then, with respect to the decomposition $H = \bigoplus_{j=1}^{\infty} \ker(N - \lambda_j)$, N has the operator matrix representation

$$N = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \end{bmatrix}.$$

Let $[S_{ij}]$ and $[X_{ij}]$ be the matrix representations of S and X with respect to the above decomposition of H. Then $NX - XN = [(\lambda_i - \lambda_j)X_{ij}]$, and in view of the assumption $S \in \{N\}'$ we have $S_{ij} = 0$ for $i \neq j$. Therefore,

$$NX - XN + S = \begin{bmatrix} S_{11} & & * \\ & S_{22} & \\ & & \ddots \\ * & & \end{bmatrix}.$$

Since $\delta_N(X) + S \in J_{|||\cdot|||}$ and since the norm of an operator matrix always dominates the norm of its diagonal part (see [9, p. 82]), it follows that

$$S \in J_{|||\cdot|||} \text{ and } |||\delta_N(X) + S||| \geq |||S|||.$$

Lemma 2. *Let $N \in B(H)$ be normal, and set $H_1 = \bigvee_{\lambda\in\mathbb{C}} \ker(N - \lambda)$. If $S \in \{N\}'$ and there is an $X \in B(H)$ such that $\delta_N(X) + S \in C_\infty$, then H_1 reduces S and $S|H_1^\perp = 0$.*

Proof. Since N is normal, H_1 reduces N and $N|H_1$ is a diagonal operator. By Fuglede's theorem (see [10, p. 104]) $S^* \in \{N\}'$, so H_1 also reduces S. Let

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

on $H = H_1 \oplus H_2$, where $H_2 = H_1^\perp$. The assumption $\delta_N(X) + S \in C_\infty$ implies $\delta_{N_2}(X_{22}) + S_2 \in C_\infty$. Anderson's result (2) (applied to the Calkin algebra $B(H_2)/C_\infty$) insures that $S_2 \in C_\infty$. Since the normal operator N_2 has no eigenvalues and since the compact selfadjoint operator $S_2^*S_2$ belongs to $\{N_2\}'$, it follows that $S_2^*S_2 = 0$. Hence $S_2 = 0$, as desired.

Now we are in a position to prove the main result of this paper.

Theorem 1. *Let $N \in B(H)$ be normal, $S \in \{N\}'$, and $X \in B(H)$. If $\delta_N(X) + S \in J_{|||\cdot|||}$, then $S \in J_{|||\cdot|||}$ and*

$$|||\delta_N(X) + S||| \geq |||S|||. \tag{8}$$

Proof. Since $\delta_N(X) + S \in J_{|||\cdot|||} \subseteq C_\infty$, it follows by Lemma 2 that on $H = H_1 \oplus H_1^\perp$,

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \text{ and } S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $H_1 = \bigvee_{\lambda\in\mathbb{C}} \ker(N-\lambda)$. If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

on $H = H_1 \oplus H_1^\perp$, then

$$\delta_N(X) + S = \begin{bmatrix} \delta_{N_1}(X_{11}) + S_1 & * \\ * & * \end{bmatrix}.$$

Since $\delta_N(X) + S \in J_{|||\cdot|||}$, it follows that $\delta_{N_1}(X_{11}) + S_1 \in J_{|||\cdot|||}$. But N_1 is diagonal and $S_1 \in \{N_1\}'$. Thus, by Lemma 1, $S_1 \in J_{|||\cdot|||}$ and $|||\delta_{N_1}(X_{11}) + S_1||| \geq |||S_1|||$. Consequently, $S \in J_{|||\cdot|||}$ and $|||\delta_N(X) + S||| \geq |||\delta_{N_1}(X_{11}) + S_1||| \geq |||S_1||| = |||S|||$.

At the end of this section we use a familiar device of considering 2×2 operator matrices to extend Theorem 1 to generalized normal derivations.

Corollary 1. *Let $N, M, S \in B(H)$ such that N and M are normal and $NS = SM$. If $X \in B(H)$ such that $\delta_{N,M}(X) + S \in J_{|||\cdot|||}$, then $S \in J_{|||\cdot|||}$ and*

(9) $$|||\delta_{N,M}(X) + S||| \geq |||S|||.$$

Proof. On $H \oplus H$ consider the operators L, T, and Y defined as

$$L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then L is normal, $T \in \{L\}'$, and

$$\delta_L(Y) + T = \begin{bmatrix} 0 & \delta_{A,B}(X) \quad + \quad S \\ 0 & 0 \end{bmatrix}.$$

Thus by Theorem 1 applied to the operators L, T, and Y we have $T \in J_{|||\cdot|||}$ and $|||\delta_L(Y)+T||| \geq |||T|||$. Therefore $S \in J_{|||\cdot|||}$ and $|||\delta_{A,B}(X)+S||| \geq |||S|||$, as desired.

3. Nonnormal derivations

Extensions of (3) to certain subnormal operators have been given in [11, Theorems 2 and 3]. In the same vein we devote this section to the extension of the results in §2 to classes of operators larger than that of normal operators.

Recall that an operator $A \in B(H)$ is called dominant (see [15]) if

(10) $$\operatorname{ran}(A-\lambda) \subseteq \operatorname{ran}(A-\lambda)^* \quad \text{for all } \lambda \in \mathbb{C}.$$

In view of [6], A is dominant if and only if for each $\lambda \in \mathbb{C}$ there exists a constant M_λ such that

(11) $$\|(A-\lambda)^* f\| \leq M_\lambda \|(A-\lambda) f\| \quad \text{for all } f \in H.$$

If there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$, then A is called M-hyponormal. If $M = 1$, then A is hyponormal.

Our promised generalization of (9) can be stated as follows.

Theorem 2. *Let $A, B, S \in B(H)$ such that A is dominant, B^* is M-hyponormal, and $AS = SB$. If $X \in B(H)$ such that $\delta_{A,B}(X) + S \in J_{|||\cdot|||}$, then $S \in J_{|||\cdot|||}$ and*

$$|||\delta_{A,B}(X) + S||| \geq |||S|||. \tag{12}$$

Proof. Since the pair (A, B) satisfies the Fuglede-Putnam property, it follows (see [16] or [19]) that $\overline{\operatorname{ran} S}$ (the closure of $\operatorname{ran} S$) reduces A, $\ker^\perp S$ (the orthogonal complement of $\ker S$) reduces B, and $A|\overline{\operatorname{ran} S}$ and $B|\ker^\perp S$ are unitarily equivalent normal operators. Then, with respect to the orthogonal decompositions $H = \overline{\operatorname{ran} S} \oplus (\overline{\operatorname{ran} S})^\perp$ and $H = \ker^\perp S \oplus \ker S$, A and B can be respectively represented as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Now assume that the operators $S, X\colon \ker^\perp S \oplus \ker S \to \overline{\operatorname{ran} S} \oplus (\overline{\operatorname{ran} S})^\perp$ have the matrix representations

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Then A_1 and B_1 are normal, and $A_1 S_1 = S_1 B_1$.

Applying Corollary 1 to the operators A_1, B_1, S_1, and X_1 we see that $S_1 \in J_{|||\cdot|||}$. Hence $S \in J_{|||\cdot|||}$ and

$$\begin{aligned} |||\delta_{A,B}(X) + S||| &= \left|\left|\left| \begin{bmatrix} \delta_{A_1,B_1}(X_1) + S_1 & * \\ * & * \end{bmatrix} \right|\right|\right| \\ &\geq |||\delta_{A_1,B_1}(X_1) + S_1||| \geq |||S_1||| = |||S|||, \end{aligned}$$

which completes the proof of the theorem.

The usual operator norm version of (12) has been obtained by Elalami [7, Theorem 4.1] using a different method.

We would like to point out here that in view of [16] Theorem 2 is still valid for any pair of operators (A, B) which satisfies the Fuglede-Putnam property, that is, $A^*S = SB^*$ whenever $AS = SB$, where $S \in B(H)$. For several such pairs, the reader is referred to [4] and references therein.

A closer look at the proof of Theorem 2 (see also [5, Theorem 1]) leads us to show that if (A, B) satisfies the Fuglede-Putnam property and if $S \in C_2$ such that $AS = SB$, then for all $X \in B(H)$ we have

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2. \tag{13}$$

This Hilbert space orthogonality result strengthens (3) and [11, Theorem 3].

It has been shown in [11, Theorem 4] that if $A \in B(H)$ is a cyclic subnormal operator and if $S \in C_2 \cap \{A\}'$, then for all $X \in B(H)$ we have

$$\|\delta_A(X) + S\|_2^2 = \|\delta_A(X)\|_2^2 + \|S\|_2^2. \tag{14}$$

In the same direction, it should be noted that the proof of Theorem 2 can be modified to insure that if $A \in B(H)$ is a cyclic subnormal operator and $S \in J_{|||\cdot|||} \cap \{A\}'$, then for all $X \in B(H)$ we have

$$|||\delta_A(X) + S||| \geq |||S|||. \tag{15}$$

To verify (15) we need only show that $\overline{\operatorname{ran} S}$ reduces A and $A|\overline{\operatorname{ran} S}$ is normal, for then we can follow the arguments in the proof of Theorem 2. Since $S \in \{A\}'$ and A is a cyclic subnormal operator, it follows by Yoshino's result [18] that S is also subnormal. This, together with the assumption $S \in J_{|||\cdot|||} \subseteq C_\infty$, implies that S is in fact normal. Consequently $S \in \{A, A^*\}'$, and so $\overline{\operatorname{ran} S}$ reduces A. If $T = AS^*$, then $T \in \{A\}'$ and $T^*T - TT^* = S(A^*A - AA^*)S^* \geq 0$ (because $A^*A - AA^* \geq 0$). Thus T is a compact hyponormal operator, and hence T is normal. Now we have $AA^*S = ASA^* = AT^* = T^*A = SA^*A = A^*AS$, and so $A|\overline{\operatorname{ran} S}$ is normal.

In [8] an example is given to show that the cyclicity assumption on A is necessary for (14) to hold. This gives an affirmative answer to a question raised in [11]. The following example, which will also be used later in the paper, is simpler and shorter than the one given in [8].

Example. Let U be the unilateral shift operator of multiplicity one. On $H \oplus H$, let $A = \left[\begin{smallmatrix} U & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $S = \left[\begin{smallmatrix} 0 & 0 \\ P & 0 \end{smallmatrix}\right]$, and $X = \left[\begin{smallmatrix} 0 & 0 \\ Q & 0 \end{smallmatrix}\right]$, where $P = 1 - UU^*$ and $Q = PU^*$. Then A is a noncyclic subnormal operator, $S \in \{A\}'$, and $\delta_A(X) + S = 0$, yet $|||\delta_A(X)||| = |||S||| = 1$ for every unitarily invariant norm $|||\cdot|||$.

This example also indicates that Anderson's result (2) cannot be extended to derivations induced by subnormal operators. However, if we require S to be normal, then in this case (2) works for several classes of operators. The list includes compact operators, dominant operators, quasinilpotent operators with positive real parts, and operators A for which $p(A) = 0$ for some quadratic polynomial p (see [7, 17]).

Another interesting class of operators for which (2) is true when S is normal is the class of operators A such that A^*A and $A + A^*$ commute. It is well known that this class enjoys the property that $\|A\| = r(A)$ (the spectral radius of A) (see [3]). Hence it is elementary to verify that (see [10, p. 130]) for all $X \in B(H)$ and all $\lambda \in \mathbb{C}$ we have

$$\|\delta_A(X) + \lambda\| \geq |\lambda|. \tag{16}$$

Based on (16) and the spectral theorem for normal operators it can be shown that if $A, S \in B(H)$ such that A^*A commutes with $A + A^*$, S is normal, and $S \in \{A\}'$, then for all $X \in B(H)$ we have

$$\|\delta_A(X) + S\| \geq \|S\|. \tag{17}$$

To prove (17) we first assume that S is a normal operator with finite spectrum. Then we use a continuity argument to establish the general case.

Finally, we remark that the example presented above shows that (2) fails to hold for an arbitrary (not necessarily normal) operator S in the commutant of A.

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