NORMAL DERIVATIONS IN NORM IDEALS

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Abstract. We establish the orthogonality of the range and the kernel of a normal derivation with respect to the unitarily invariant norms associated with norm ideals of operators. Related orthogonality results for certain nonnormal derivations are also given.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex separable Hilbert space $H$. For operators $A, B$ in $B(H)$, the generalized derivation $\delta_{A,B}$ as an operator on $B(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB \quad \text{for all } X \in B(H).$$

When $A = B$, we simply write $\delta_A$ for $\delta_{A,A}$. If $N$ is a normal operator in $B(H)$, then $\delta_N$ is said to be a normal derivation.

In his investigation of normal derivations, Anderson [1, Theorem 1.7] proved that if $N$ and $S$ are operators in $B(H)$ such that $TN$ is normal and $NS = SN$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\| \geq \|S\|,$$

where $\| \cdot \|$ is the usual operator norm. Thus in the sense of [1, Definition 1.2], inequality (2) says that the range of $\delta_N$ is orthogonal to the kernel of $\delta_N$, which is just the commutant $\{N\}'$ of $N$.

It has been shown in [11, Theorem 1] that if $N$ and $S$ are operators in $B(H)$ such that $N$ is normal, $S$ is a Hilbert-Schmidt operator, and $S \in \{N\}'$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2,$$

where $\| \cdot \|_2$ is the Hilbert-Schmidt norm. Thus in the usual Hilbert space sense, the Hilbert-Schmidt operators in the range of $\delta_N$ are orthogonal to those in the kernel of $\delta_N$.

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It has also been shown recently in [12, Theorem 3.2] that if \( N \) and \( S \) are operators in \( B(H) \) such that \( N \) is normal and \( S \) belongs to some Schatten \( p \)-class \( C_p \) with \( 1 \leq p \leq \infty \) and \( S \in \{N\}' \), then for all \( X \in B(H) \)
\[
\|\delta_N(X) + S\|_p \geq \|S\|_p.
\]
The usual operator norm, the Hilbert-Schmidt norm, and the Schatten \( p \)-norms are only examples of a large family of unitarily invariant (or symmetric) norms on \( B(H) \).

The purpose of this paper is to investigate the orthogonality of the range and the kernel of a normal derivation with respect to the wider class of unitarily invariant norms on \( B(H) \). Derivations induced by certain nonnormal operators will also be discussed.

In §2 we will use a completely different analysis to extend (4) to all unitarily invariant norms defined on norm ideals of compact operators in \( B(H) \). Extensions of this result to certain nonnormal operators will be the main theme of §3, in which we will treat derivations of the form \( \delta_{A,B} \), where \( A \) is a dominant operator and \( B^* \) is \( M \)-hyponormal. Moreover we will discuss the validity of (2) for various classes of derivations at the expense of requiring that \( S \) is normal. A relevant example will also be presented.

Recall that each unitarily invariant norm \( \||| \cdot ||| \) is defined on a natural subclass \( J_{||| \cdot |||} \) of \( B(H) \) called the norm ideal associated with the norm \( ||| \cdot ||| \) and satisfies the invariance property \( \||| UAV ||| = |||A||| \) for all \( A \in J_{||| \cdot |||} \) and for all unitary operators \( U, V \in B(H) \). While the usual operator norm \( || \cdot || \) is defined on all of \( B(H) \), the other unitarily invariant norms are defined on norm ideals contained in the ideal of compact operators in \( B(H) \). Given any compact operator \( A \in B(H) \), denote by \( s_1(A) \geq s_2(A) \geq \cdots \) the singular values of \( A \), i.e., the eigenvalues of \( |A| = (A^*A)^{1/2} \). There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if \( \||| \cdot ||| \) is a unitarily invariant norm, then there is a unique symmetric gauge function \( \Phi \) such that
\[
\||| A ||| = \Phi(\{s_j(A)\})
\]
for all \( A \in J_{||| \cdot |||} \).

For \( 1 \leq p \leq \infty \), define
\[
\|A\|_p = \left( \sum_j s_j^p(A) \right)^{1/p},
\]
where, by convention, \( \|A\|_\infty = s_1(A) \) is the usual operator norm of the compact operator \( A \). These unitarily invariant norms are the well-known Schatten \( p \)-norms associated with the Schatten \( p \)-classes \( C_p \), \( 1 \leq p \leq \infty \). Hence \( C_1, C_2 \), and \( C_\infty \) are the trace class, the Hilbert-Schmidt class, and the class of compact operators, respectively. For good accounts on the theory of norm ideals and their associated unitarily invariant norms, the reader is referred to [9], [13], or [14] (see also [2] and references therein).

2. Normal derivations

In this section we present our main result of this paper. This result asserts that if \( N \) is a normal operator in \( B(H) \), then with respect to any unitarily
invariant norm $||| \cdot |||$, $\text{ran} \delta_N \cap J_{||| \cdot |||}$ is orthogonal to $\text{ker} \delta_N \cap J_{||| \cdot |||}$, where $\text{ran} \delta_N$ and $\text{ker} \delta_N$ are the range and the kernel of $\delta_N$, respectively.

To accomplish our goal we need two lemmas.

**Lemma 1.** Let $N \in B(H)$ be diagonal (normal with pure point spectrum), $S \in \{N\}'$, and $X \in B(H)$. If $\delta_N(X) + S \in J_{||| \cdot |||}$, then $S \in J_{||| \cdot |||}$ and

$$||| \delta_N(X) + S ||| \geq |||S|||.$$  

**Proof.** Let $N$ have the distinct eigenvalues $\lambda_1, \lambda_2, \ldots$. Then, with respect to the decomposition $H = \bigoplus_{j=1}^{\infty} \text{ker}(N - \lambda_j)$, $N$ has the operator matrix representation

$$N = \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \ddots & 0 \\ \end{bmatrix}.$$ 

Let $[S_{ij}]$ and $[X_{ij}]$ be the matrix representations of $S$ and $X$ with respect to the above decomposition of $H$. Then $NX - XN = \sum_{i,j} (X_{ij} - X_{ji}) X_{ij}$, and in view of the assumption $S \in \{N\}'$ we have $S_{ij} = 0$ for $i \neq j$. Therefore,

$$SX - XS + S = \begin{bmatrix} S_{11} & \ast \\ \ast & \ddots \\ \end{bmatrix}.$$ 

Since $\delta_N(X) + S \in J_{||| \cdot |||}$ and since the norm of an operator matrix always dominates the norm of its diagonal part (see [9, p. 82]), it follows that

$S \in J_{||| \cdot |||}$ and $||| \delta_N(X) + S ||| \geq |||S|||$.

**Lemma 2.** Let $N \in B(H)$ be normal, and set $H_1 = \bigvee_{\lambda \in \mathbb{C}} \text{ker}(N - \lambda)$. If $S \in \{N\}'$ and there is an $X \in B(H)$ such that $\delta_N(X) + S \in C_\infty$, then $H_1$ reduces $S$ and $S|H_1^\perp = 0$.

**Proof.** Since $N$ is normal, $H_1$ reduces $N$ and $N|H_1$ is a diagonal operator. By Fuglede's theorem (see [10, p. 104]) $S^* \in \{N\}'$, so $H_1$ also reduces $S$. Let

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \\ \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \\ \end{bmatrix}, \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ \end{bmatrix}$$

on $H = H_1 \oplus H_2$, where $H_2 = H_1^\perp$. The assumption $\delta_N(X) + S \in C_\infty$ implies $\delta_{N_1}(X_{22}) + S_2 \in C_\infty$. Anderson's result (2) (applied to the Calkin algebra $B(H_2)/C_\infty$) insures that $S_2 \in C_\infty$. Since the normal operator $N_2$ has no eigenvalues and since the compact selfadjoint operator $S_2^* S_2$ belongs to $\{N_2\}'$, it follows that $S_2^* S_2 = 0$. Hence $S_2 = 0$, as desired.

Now we are in a position to prove the main result of this paper.

**Theorem 1.** Let $N \in B(H)$ be normal, $S \in \{N\}'$, and $X \in B(H)$. If $\delta_N(X) + S \in J_{||| \cdot |||}$, then $S \in J_{||| \cdot |||}$ and

$$||| \delta_N(X) + S ||| \geq |||S|||.$$  

**Proof.** Since $\delta_N(X) + S \in J_{||| \cdot |||} \subseteq C_\infty$, it follows by Lemma 2 that on $H = H_1 \oplus H_1^\perp$,

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \\ \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \\ \end{bmatrix},$$

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where $H_1 = \bigvee_{\lambda \in \mathbb{C}} \ker (N - \lambda)$. If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

on $H = H_1 \oplus H_1^\perp$, then

$$\delta_N(X) + S = \begin{bmatrix} \delta_{N_1}(X_{11}) + S_1 & * \\ * & * \end{bmatrix}.$$

Since $\delta_N(X) + S \in J_{|| \cdot ||}$, it follows that $\delta_{N_1}(X_{11}) + S_1 \in J_{|| \cdot ||}$. But $N_1$ is diagonal and $S_1 \in \{N_1\}'$. Thus, by Lemma 1, $S_1 \in J_{|| \cdot ||}$ and $||\delta_{N_1}(X_{11}) + S_1|| \geq ||S_1||$. Consequently, $S \in J_{|| \cdot ||}$ and $||\delta_N(X) + S|| \geq ||\delta_{N_1}(X_{11}) + S_1|| \geq ||S_1|| = ||S||$.

At the end of this section we use a familiar device of considering $2 \times 2$ operator matrices to extend Theorem 1 to generalized normal derivations.

**Corollary 1.** Let $N, M, S \in B(H)$ such that $N$ and $M$ are normal and $NS = SM$. If $X \in B(H)$ such that $\delta_{N,M}(X) + S \in J_{|| \cdot ||}$, then $S \in J_{|| \cdot ||}$ and

$$||\delta_{N,M}(X) + S|| \geq ||S||.$$  

**Proof.** On $H \oplus H$ consider the operators $L, T$, and $Y$ defined as

$$L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \quad T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then $L$ is normal, $T \in \{L\}'$, and

$$\delta_L(Y) + T = \begin{bmatrix} 0 & \delta_{A,B}(X) + S \\ 0 & 0 \end{bmatrix}.$$

Thus by Theorem 1 applied to the operators $L, T$, and $Y$ we have $T \in J_{|| \cdot ||}$ and $||\delta_L(Y) + T|| \geq ||T||$. Therefore $S \in J_{|| \cdot ||}$ and $||\delta_{A,B}(X) + S|| \geq ||S||$, as desired.

### 3. Nonnormal derivations

Extensions of (3) to certain subnormal operators have been given in [11, Theorems 2 and 3]. In the same vein we devote this section to the extension of the results in §2 to classes of operators larger than that of normal operators.

Recall that an operator $A \in B(H)$ is called dominant (see [15]) if

$$(10) \quad \text{ran}(A - \lambda) \subseteq \text{ran}(A - \lambda)^* \quad \text{for all } \lambda \in \mathbb{C}. $$

In view of [6], $A$ is dominant if and only if for each $\lambda \in \mathbb{C}$ there exists a constant $M_\lambda$ such that

$$(11) \quad \| (A - \lambda)^* f \| \leq M_\lambda \| (A - \lambda) f \| \quad \text{for all } f \in H. $$

If there is a constant $M$ such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$, then $A$ is called $M$-hyponormal. If $M = 1$, then $A$ is hyponormal.

Our promised generalization of (9) can be stated as follows.
**Theorem 2.** Let $A, B, S \in B(H)$ such that $A$ is dominant, $B^*$ is $M$-hyponormal, and $AS = SB$. If $X \in B(H)$ such that $\delta_{A,B}(X) + S \in J_{\|\cdot\|}$, then

$$\|\|\delta_{A,B}(X) + S\|\| \geq \|\|S\||\|.$$  

**Proof.** Since the pair $(A, B)$ satisfies the Fuglede-Putnam property, it follows (see [16] or [19]) that $\text{ran } S$ (the closure of $\text{ran } S$) reduces $A$, $\ker^\perp S$ (the orthogonal complement of $\ker S$) reduces $B$, and $A|\text{ran } S$ and $B|\ker^\perp S$ are unitarily equivalent normal operators. Then, with respect to the orthogonal decompositions $H = \text{ran } S \oplus (\text{ran } S)^\perp$ and $H = \ker^\perp S \oplus \ker S$, $A$ and $B$ can be respectively represented as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$ 

Now assume that the operators $S, X : \ker^\perp S \oplus \ker S \to \text{ran } S \oplus (\text{ran } S)^\perp$ have the matrix representations

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$ 

Then $A_1$ and $B_1$ are normal, and $A_1 S_1 = S_1 B_1$.

Applying Corollary 1 to the operators $A_1, B_1, S_1$, and $X_1$ we see that $S_1 \in J_{\|\cdot\|}$. Hence $S \in J_{\|\cdot\|}$ and

$$\|\|\delta_{A,B}(X) + S\|\| = \|\|\begin{bmatrix} \delta_{A_1,B_1}(X_1) + S_1 & * \\ * & * \end{bmatrix}\|\|,$$

$$\geq \|\|\delta_{A_1,B_1}(X_1) + S_1\|\| \geq \|\|S_1\||\| = \|\|S\||\|,$$

which completes the proof of the theorem.

The usual operator norm version of (12) has been obtained by Elalami [7, Theorem 4.1] using a different method.

We would like to point out here that in view of [16] Theorem 2 is still valid for any pair of operators $(A, B)$ which satisfies the Fuglede-Putnam property, that is, $A^* S = SB^*$ whenever $AS = SB$, where $S \in B(H)$. For several such pairs, the reader is referred to [4] and references therein.

A closer look at the proof of Theorem 2 (see also [5, Theorem 1]) leads us to show that if $(A, B)$ satisfies the Fuglede-Putnam property and if $S \in C_2$ such that $AS = SB$, then for all $X \in B(H)$ we have

$$\|\|\delta_{A,B}(X) + S\|\|^2 = \|\|\delta_{A,B}(X)\|\|^2 + \|\|S\|\|^2.$$  

This Hilbert space orthogonality result strengthens (3) and [11, Theorem 3].

It has been shown in [11, Theorem 4] that if $A \in B(H)$ is a cyclic subnormal operator and if $S \in C_2 \cap \{A\}'$, then for all $X \in B(H)$ we have

$$\|\|\delta_A(X) + S\|\|^2 = \|\|\delta_A(X)\|\|^2 + \|\|S\|\|^2.$$  

In the same direction, it should be noted that the proof of Theorem 2 can be modified to insure that if $A \in B(H)$ is a cyclic subnormal operator and $S \in J_{\|\cdot\|} \cap \{A\}'$, then for all $X \in B(H)$ we have

$$\|\|\delta_A(X) + S\|\| \geq \|\|S\||\|. $$
To verify (15) we need only show that \( \text{ran} S \) reduces \( A \) and \( \text{ran} S \) is normal, for then we can follow the arguments in the proof of Theorem 2. Since \( S \in \{A\}' \) and \( A \) is a cyclic subnormal operator, it follows by Yoshino's result \([18]\) that \( S \) is also subnormal. This, together with the assumption \( S \in J_{\| \cdot \|} \subseteq C_\infty \), implies that \( S \) is in fact normal. Consequently \( S \in \{A, A^*\}' \), and so \( \text{ran} S \) reduces \( A \). If \( T = AS^* \), then \( T \in \{A\}' \) and \( T^* - TT^* = S(A^*A - AA^*)S^* \geq 0 \) (because \( A^*A - AA^* \geq 0 \)). Thus \( T \) is a compact subnormal operator, and hence \( T \) is normal. Now we have \( AA^*S = ASA^* = AT^* = T^*A = SA^*A = A^*AS \), and so \( A|\text{ran} S \) is normal.

In \([8]\) an example is given to show that the cyclicity assumption on \( A \) is necessary for (14) to hold. This gives an affirmative answer to a question raised in \([11]\). The following example, which will also be used later in the paper, is simpler and shorter than the one given in \([8]\).

**Example.** Let \( U \) be the unilateral shift operator of multiplicity one. On \( H \oplus H \), let \( A = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \), \( S = \begin{bmatrix} 0 & 0 \\ P & 0 \end{bmatrix} \), and \( X = \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix} \), where \( P = 1 - UU^* \) and \( Q = PU^* \). Then \( A \) is a noncyclic subnormal operator, \( S \in \{A\}' \), and \( \delta_A(X)+S = 0 \), yet \( \|\delta_A(X)\| = \|S\| = 1 \) for every unitarily invariant norm \( \| \cdot \| \).

This example also indicates that Anderson's result \((2)\) cannot be extended to derivations induced by subnormal operators. However, if we require \( S \) to be normal, then in this case \((2)\) works for several classes of operators. The list includes compact operators, dominant operators, quasinilpotent operators with positive real parts, and operators \( A \) for which \( p(A) = 0 \) for some quadratic polynomial \( p \) (see \([7, 17]\)).

Another interesting class of operators for which \((2)\) is true when \( S \) is normal is the class of operators \( A \) such that \( A^*A \) and \( A + A^* \) commute. It is well known that this class enjoys the property that \( \|A\| = r(A) \) (the spectral radius of \( A \)) (see \([3]\)). Hence it is elementary to verify that (see \([10, p. 130]\)) for all \( X \in B(H) \) and all \( \lambda \in \mathbb{C} \) we have

\[
\|\delta_A(X) + \lambda\| \geq |\lambda|.
\]

Based on \((16)\) and the spectral theorem for normal operators it can be shown that if \( A, S \in B(H) \) such that \( A^*A \) commutes with \( A + A^* \), \( S \) is normal, and \( S \in \{A\}' \), then for all \( X \in B(H) \) we have

\[
\|\delta_A(X) + S\| \geq \|S\|.
\]

To prove \((17)\) we first assume that \( S \) is a normal operator with finite spectrum. Then we use a continuity argument to establish the general case.

Finally, we remark that the example presented above shows that \((2)\) fails to hold for an arbitrary (not necessarily normal) operator \( S \) in the commutant of \( A \).

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**References**


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