ON THE MULTIPLE POINTS
OF CERTAIN MEROMORPHIC FUNCTIONS

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Abstract. We show that if \( f \) is transcendental and meromorphic in the plane and \( T(r, f) = o(\log r)^2 \), then \( f \) has infinitely many critical values. This is sharp. Further, we apply a result of Eremenko to show that if \( f \) is meromorphic of finite lower order in the plane and \( N(r, 1/f') = o(T(r, f'/f)) \), then \( f(z) = \exp(az + b) \) or \( f(z) = (az + b)^{-n} \) with \( a \) and \( b \) constants and \( n \) a positive integer.

1. Introduction

If \( g \) is a function transcendental and meromorphic in the plane, then the term
\[
N_1(r, g) = N(r, g) - \overline{N}(r, g) + N(r, 1/g'),
\]
in which the counting functions are defined as in [11, Chapter 2], counts the multiple points of \( g \). The following has been proved by Eremenko.

**Theorem A** [5]. Let \( g \) be transcendental and meromorphic in the plane of finite lower order such that \( \delta(\infty, g) = 0 \) and \( N_1(r, g) = o(T(r, g)) \). Then there exist an integer \( 2 \rho \geq 2 \) and continuous functions \( L_1(r) \) and \( L_2(r) \) such that \( L_1(ct) = L_1(t)(1 + o(1)) \) and \( L_2(ct) = L_2(t) + o(1) \) as \( t \to +\infty \), uniformly for \( 1 \leq c \leq 2 \), and such that
\[
-\log|g'(re^{i\theta})| = \pi \rho L_1(\rho)|\cos(\rho(\theta - L_2(\rho)))| + o(\rho L_1(\rho))
\]
as \( r \to +\infty \), uniformly in \( \theta \), \( 0 \leq \theta \leq 2\pi \), provided that \( re^{i\theta} \) lies outside an exceptional set \( C_0 \) of discs \( B(z_k, r_k) \) with the property that if \( R \) is large, then the sum of the radii \( r_k \) of the discs \( B(z_k, r_k) \) for which \( |z_k| < R \) is \( o(R) \). Further, \( \sum_{a\in\mathbb{C}} \delta(a, g) = 2 \) and \( T(r, g) = (1 + o(1))\rho L_1(r) \).

It follows from Theorem A that a transcendental meromorphic function \( g \) of order less than 1 cannot satisfy \( N_1(r, g) = o(T(r, g)) \) and so must have multiple points (Shea [19] had earlier proved this when \( g \) has order less than \( 1/2 \)). Infinitely many of these multiple points must be zeros of \( g' \), as is shown by the following result from [6].
Theorem B. Suppose that \( g \) is transcendental and meromorphic in the plane with \( T(r, g) = o(r) \). Then \( g' \) has infinitely many zeros.

Further, if \( g \) is transcendental meromorphic with \( T(r, g) = o(r^{1/2}) \) or transcendental entire with \( T(r, g) = o(r) \), then \( g'/g \) must have zeros. These assertions are proved in [3] and are shown there to be sharp.

This suggests the question as to whether a growth condition on a transcendental meromorphic function \( f \) forces \( f \) to have infinitely many critical values, that is, values taken by \( f \) at multiple points of \( f \). If \( T(r, f) = o(r^{1/2}) \) and \( f \) is transcendental with only finitely many poles, it is easily seen from the discussion in [18, pp. 269-272] that \( \infty \) must be an accumulation point of critical values of \( f \), for otherwise the inverse function \( f^{-1} \) would have a logarithmic singularity at \( \infty \) and, if \( R \) is large, would exist a simply-connected unbounded component \( U \) of the set \( \{ z \in \mathbb{C} : |f(z)| > R \} \), with \( |f(z)| = R \) on the boundary of \( U \), which contradicts the \( \cos \pi \rho \) theorem [11, p. 119]. Corresponding to this remark is the obvious example \( \cos(\sqrt{z}) \).

The above observation and example also appear in [2], of which the author became aware after the first draft of the present paper was written. Among other results in [2] concerning asymptotic and critical values of meromorphic functions, it is shown (Corollary 3) that if the transcendental meromorphic function \( f \) has finite order \( \rho \) and only finitely many critical values, then the number of asymptotic values of \( f \) is at most \( 2\rho \).

While a transcendental entire function always has \( \infty \) as an asymptotic value, by the classical theorem of Iversen [18], meromorphic functions need not have any asymptotic values at all. Bank and Kaufman [1] (see also [13, Chapter 11]) proved the existence of a function \( f \) transcendental and meromorphic in the plane with \( T(r, f) = O(\log r)^2 \), satisfying the differential equation

\[
(z^2 - 4)(f'(z))^2 = 4(f(z) - e_1)(f(z) - e_2)(f(z) - e_3),
\]

in which the \( e_i \) are distinct complex numbers, and this function \( f \) clearly has just 4 critical values. This example is obtained from the Weierstrass doubly periodic function. For smaller growth, we prove the following theorem, the proof of which is based on a combination of representations for the function in annuli with the Riemann-Hurwitz formula.

Theorem 1. If \( f \) is transcendental and meromorphic in the plane with \( T(r, f) = o(\log r)^2 \), then \( f \) has infinitely many critical values.

Our second result is a fairly straightforward application of Theorem A, coupled with a variant of a method of Mues from [17]. It was proved in [14] that if \( f \) is meromorphic in the plane and \( f \) and \( f'' \) have no zeros, then \( f(z) = \exp(a z + b) \) or \( f(z) = (a z + b)^{-n} \) with \( a \) and \( b \) constants and \( n \) a positive integer. This proved a conjecture of Hayman [10, 12], the case where \( f \) has finite order having been settled by Mues in [17]. The same conclusion holds if \( f \) is meromorphic in the plane and \( N(r, 1/ff^{(k)}) = o(T(r, f'/f)) \) for some \( k \geq 3 \) [8, Theorem 2; see also 7, 9]. We prove here the following result.

Theorem 2. Suppose that \( f \) is meromorphic of finite lower order in the plane and that \( a_1 \) and \( a_0 \) are rational functions such that the differential equation

\[
y'' + a_1 y' + a_0 y = 0
\]
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has linearly independent rational solutions \( f_1 \) and \( f_2 \). If \( F(z) = f''(z) + a_1(z)f'(z) + a_0(z)f(z) \), then \( f'/f \) is rational and \( f \) and \( F \) have no zeros.

**Corollary.** If \( f \) is meromorphic of finite lower order in the plane and

\[
N(r, 1/fF) = o(T(r, f'/f)),
\]

then \( f(z) = \exp(az + b) \) or \( f(z) = (az + b)^{-n} \) with \( a \) and \( b \) constants and \( n \) a positive integer.

The corollary follows at once from Theorem 2 using [11, p. 76]. Note that the assumption that (1.1) has a rational fundamental solution set implies that \( u_j(z) = O(|z|^{-2}) \) as \( z \to \infty \) and that larger coefficients cannot be allowed in general, as the example \( g(z) = \sec(\sqrt{z}) \), \( G(z) = g''(z) + (1/2z)g'(z) + (1/4z)g(z) = g^3(z)/2z \), shows. In [14] and [15], the author determined all functions \( f \) meromorphic in the plane such that \( f \) and \( f'' + a_1f' + a_0f \) have only finitely many zeros, where \( a_1 \) and \( a_0 \) are rational. It seems possible that the conclusion of Theorem 2 would be true without any assumption on the growth of \( f \) and with \( a_1 \) and \( a_0 \) any rational functions satisfying \( a_j(z) = O(|z|^{-2}) \) (in which case (1.1) might not have solutions meromorphic in a neighbourhood of infinity), but the present proof, which consists of applying Theorem A to

\[
H(z) = \frac{f_1'(z) - (f'(z)/f(z))f_1(z)}{f_2'(z) - (f'(z)/f(z))f_2(z)},
\]

requires \( H \) to be meromorphic in the plane of finite lower order.

2. Preliminaries

A key role in the proof of Theorem 1 is played by the Riemann-Hurwitz formula (see [20, Chapter 1]): Suppose that \( D \) and \( G \) are bounded domains of connectivity \( m \) and \( n \) respectively and that \( f: D \to G \) is an analytic function with the property that, for any sequence \( (z_k) \) in \( D \), \( z_k \) tends to the boundary \( \partial D \) as \( k \to \infty \) (in the sense that if \( K \) is a compact subset of \( D \), then \( z_k \in D\setminus K \) for all large enough \( k \)) if and only if \( f(z_k) \) tends to \( \partial G \). Then there exists a positive integer \( p \) such that all values \( w \) belonging to \( G \) are taken \( p \) times in \( D \), counting multiplicities, and such that \( m - 2 = p(n - 2) + r \), where \( r \) is the number of critical points of \( f \) in \( D \), that is, the number of zeros of \( f' \) in \( D \), counting multiplicities.

Suppose now that \( f \) is a function meromorphic in the plane with only finitely many critical values. If \( R \) and \( S \) are large, any bounded component of the set \( \{ z \in \mathbb{C} : R < |f(z)| < S \} \) must be doubly-connected, while any bounded component of the set \( \{ z \in \mathbb{C} : |f(z)| > S \} \) contains one (possibly multiple) pole of \( f \) and is simply-connected.

**Lemma 1.** Let \( n(t) \) be nondecreasing, integer valued, and continuous from the right such that \( n(1) = 0 \) and \( n(t) = o(\log t) \) as \( t \to +\infty \). Set

\[
h(r) = \int_1^r t \, dn(t).
\]

If \( \delta \) is a positive constant, then the set \( E(\delta) = \{ r \geq 1 : h(r) \geq \delta r \} \) has logarithmic density 0.
Proof. Let \( \chi(t) \) be the characteristic function of \( E(\delta) \), so that \( \chi(t) = 1 \) if \( t \geq 1 \) and \( t \in E(\delta) \) and \( \chi(t) = 0 \) otherwise. Then

\[
\int_1^r \chi(t)/t \, dt \leq (1/\delta) \int_1^r h(t)/t^2 \, dt = (1/\delta) \int_1^r 1/t \, dh(t) - h(r)/\delta r \\
\leq (1/\delta) \int_1^r dn(t) = o(\log r),
\]

which is precisely what is asserted in the lemma.

The next lemma is part of a special case of the lemma from [16].

**Lemma A.** Let \( m(t) \) be nondecreasing, integer-valued and continuous from the right, with \( m(1) = 0 \) and \( m(t) = O(t) \) as \( t \to +\infty \). Let \( M > 3 \) be a constant. Then there exists a set \( E_M \) of lower logarithmic density at least \( 1 - 3/M \); that is,

\[
\int_1^r \chi(t)/t \, dt > (1 - 3/M - o(1)) \log r \quad \text{as} \quad r \to +\infty,
\]

with \( \chi(t) \) the characteristic function of \( E_M \), such that, for \( r \in E_M \) and \( t \geq r \), we have \( m(t)/m(r) \leq (t/r)^4M \).

**Lemma 2.** Let \( f \) be transcendental and meromorphic in the plane with \( T(r, f) = o(\log r)^2 \). Then there exist sequences \( R_\nu \) and \( S_\nu \) tending to \( +\infty \), nonzero constants \( C_\nu \) and \( D_\nu \), and integers \( m_\nu \) and \( n_\nu \) such that for

\[
(2.1) \quad R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu
\]

we have

\[
(2.2) \quad f(z) = C_\nu z^{m_\nu}(1 + o(1))
\]

and

\[
(2.3) \quad f'(z) = D_\nu z^{n_\nu}(1 + o(1)).
\]

**Proof.** We write \( f(z) = U(z)F(z) \) and \( f'(z) = V(z)G(z) \) where \( U \) and \( V \) are rational functions and \( F \) and \( G \) satisfy \( F(0) = G(0) = 1 \) and have no zeros or poles in \( |z| \leq 1 \). We choose a small positive \( \delta \) and apply Lemma 1 with \( n(t) = n(t, 1/F) + n(t, F) + n(t, 1/G) + n(t, G) = O(T(t^2, f)/\log t) = o(\log t) \). Further, we apply Lemma 2 with \( M = 100 \) and \( m(t) = 2^n(t) \). This gives arbitrarily large \( r \) such that

\[
(2.4) \quad h(r) = \int_1^r t \, dn(t) < \delta r
\]

and, for \( t \geq r \),

\[
(2.5) \quad n(t) - n(r) \leq M_1 \log(t/r),
\]

where \( M_1 = 400/\log 2 \). Since \( n(t) \) is integer-valued, (2.4) implies that \( f \) and \( f' \) have no zeros or poles in \( \delta r \leq |z| \leq r \). Suppose that

\[
(2.6) \quad \delta^{3/4} r \leq |z| \leq \delta^{1/4} r.
\]

We write \( F(z) = F_1(z)/F_2(z) \) with the \( f_j \) entire and \( f_1(z) = \prod_{j=1}^\infty (1 - z/a_j) \), where the \( a_j \) are the zeros of \( f \) in \( 1 < |z| < \infty \), repeated according to multiplicity. For \( z \) as in (2.6) we have

\[
(2.7) \quad f_1(z) = z^{\nu(r, 1/F)} \prod_i (-1/a_j) \prod_i (1 - a_j/z) \prod_i (1 - z/a_j),
\]
in which $\prod_1$ denotes the product over all $a_j$ with $|a_j| < r$ and $\prod_2$ denotes the product over the remaining $a_j$. With $\sum_1$ defined analogously to $\prod_1$, we have, using (2.4),

$$|\prod_1(1 - a_j/z) - 1| \leq \exp(\sum_1|a_j/z|) - 1$$

$$\leq \exp(h(r)/|z|) - 1 \leq \exp(\delta r/|z|) - 1$$

$$\leq \exp(\delta^{1/4}) - 1.$$  

Further, (2.5) gives $n(t, 1/f) - n(r, 1/f) \leq M_1(\log(t/r))$ for $t \geq r$, and we have

$$|\prod_2(1 - z/a_j) - 1| \leq \exp\left(|z|\int_r^\infty \frac{1}{t} dn(t, 1/f)\right) - 1$$

$$= \exp\left(|z|\int_r^\infty [n(t, 1/f) - n(r, 1/f)] dt/t^2\right) - 1$$

$$\leq \exp\left(|z|M_1\int_r^\infty \log(t/r) dt/t^2\right) - 1$$

$$= \exp(|z|M_1/r - 1 \leq \exp(M_1\delta^{1/4}) - 1.$$  

Now if $\varepsilon > 0$ is given, we need only choose $\delta$ small enough, and (2.7), (2.8), and (2.9) then give $f_1(z) = \prod_1(-1/a_j)z^n(r, 1/f)(1 + \rho(z))$, where $|\rho(z)| < \varepsilon$ for $z$ satisfying (2.6). Estimating $f_2$ in the same way gives (2.2) and the proof of (2.3) is identical.

The following is Lemma III of [4].

**Lemma B.** Suppose that $g$ is meromorphic in $|z| < R$, $1 < r < R$, and that $I(r)$ is a measurable subset of $[0, 2\pi]$ of measure $\mu(r)$. Then

$$\frac{1}{2\pi} \int_{I(r)} \log^+ |g(re^{i\theta})| d\theta \leq 11R(R - r)^{-1}\mu(r)\left(1 + \log + \frac{1}{\mu(r)}\right) T(R, g).$$

### 3. Proof of Theorem 1

Suppose that $f$ is transcendental and meromorphic in the plane such that $T(r, f) = o(\log r)^2$ and $f$ has only finitely many critical values. By the remark in the introduction we can assume that $f$ has no Picard value. Let $R_\nu$, $S_\nu$, $C_\nu$, $D_\nu$, $m_\nu$, $n_\nu$ be as in Lemma 2. We can assume that, as $\nu \to \infty$,

$$|C_\nu|R_\nu^{m_\nu} \to \alpha, \quad 1 \leq \alpha \leq +\infty,$$

by taking a subsequence and replacing $f$ by $1/f$, if necessary. We consider a number of cases.

**Case 1.** Suppose that $\alpha = +\infty$ and $m_\nu = 0$ for infinitely many $\nu$.

Taking a further subsequence if necessary we can assume that

$$100|C_{\nu-1}|R_\nu^{m_{\nu-1}} < |C_\nu| < 100^{-1}|C_{\nu+1}|R_\nu^{m_{\nu+1}}.$$  

Take a small positive $\varepsilon$. Now (3.2) implies that if $\nu$ is large enough, the circle $|z| = R_\nu$ lies inside a bounded component of the set $\{z: |f(z) - C_\nu| < \varepsilon |C_\nu|\}$. This component contains no multiple point of $f$ and is multiply connected, by (3.2), which contradicts the Riemann-Hurwitz formula.
Case 2. Suppose that \( \alpha = +\infty \) and that \( m_\nu \neq 0 \) for all large \( \nu \).

Then (2.2) implies that the annulus \( (1/4)R_\nu < |z| < 4R_\nu \) contains a closed level curve \( \Gamma_\nu \) on which \( |F(z)| = k_\nu = |C_\nu|R_\nu^{m_\nu} \), and this level curve \( \Gamma_\nu \) must be a simple closed curve winding once around the origin. We take \( \mu < \nu \) such that \( 100k_\mu < k_\nu \) and such that the region \( U \) lying between \( \Gamma_\mu \) and \( \Gamma_\nu \) contains at least one zero of \( f \).

Let \( V_1 \) be a component of the set \( \{z \in U : |f(z)| < k_\mu\} \). Since \( |f(z)| \geq k_\mu \) on \( \Gamma_\mu \) and \( \Gamma_\nu \), we have \( |f(z)| = k_\mu \) on the boundary \( \partial V_1 \), which is contained in the closure \( \overline{U} \) of \( U \) and consists of disjoint smooth simple closed curves. Let \( \gamma_1 \) be the unique component of \( \partial V_1 \) which forms the boundary of an unbounded component of \( C\backslash V_1 \), and suppose first that the winding number \( n(\gamma_1, 0) = 0 \).

Now \( \gamma_1 \) cannot coincide with \( \Gamma_\mu \), since the interior of \( \Gamma_\mu \) is bounded, and so \( \gamma_1 \) does not meet \( \Gamma_\mu \), using the fact that \( f \) has no critical values on \( |w| = k_\mu \). Thus \( \gamma_1 \) forms part of the boundary of a component \( V^* \) of the set \( \{z \in U : k_\mu < |f(z)| < k_\nu\} \). On \( \partial V^* \) we have \( |f(z)| = k_\mu \) or \( |f(z)| = k_\nu \), and \( V^* \) must be doubly-connected, by the Riemann-Hurwitz formula. The other component \( \gamma_2 \) on \( \partial V^* \) must close in \( \overline{U} \) and cannot coincide with \( \Gamma_\nu \), since \( V^* \) is doubly-connected and since there exist points arbitrarily close to \( \Gamma_\mu \) at which \( |f(z)| < k_\mu \). Thus \( \gamma_2 \) cannot meet \( \Gamma_\nu \). Further, on \( \gamma_2 \) we have \( |f(z)| = k_\nu \), and \( \gamma_2 \) forms part of the boundary of a component \( V^{**} \) of the set \( \{z \in U : |f(z)| > k_\nu\} \), on the boundary of which \( |f(z)| = k_\nu \). The Riemann-Hurwitz formula now implies that \( V^{**} \) is simply-connected, which is a contradiction, since \( V^* \) lies in a bounded component of \( C\backslash V_1 \).

This contradiction proves that \( n(\gamma_1, 0) \neq 0 \). Thus \( \Gamma_\mu \) lies in a bounded component of \( C\backslash V_1 \), and \( \partial V_1 \) must have precisely two components \( \omega_1 \) and \( \omega_2 \) such that \( n(\omega_j, 0) \neq 0 \), and this is true for every component of the set \( \{z \in U : |f(z)| < k_\mu\} \). If these components are \( V_1, \ldots, V_p \) and if \( p > 1 \), we can assume that, for each \( j \), \( V_j \) lies in the same component of \( C\backslash V_{j+1} \) as \( \Gamma_\mu \). But then components of \( \partial V_1 \) and \( \partial V_2 \) together bound a doubly-connected region on which \( |f(z)| \geq k_\mu \). This region must contain a pole of \( f \) by the maximum principle, and the fact that it is not simply-connected contradicts the Riemann-Hurwitz formula. Therefore \( p = 1 \), which is a contradiction since we can choose \( \nu \) arbitrarily large.

Case 3. Suppose that \( \alpha \) is finite in (3.1).

If \( m_\nu \neq 0 \) for infinitely many \( \nu \), then since \( S_\nu \to \infty \) we can take a subsequence and obtain level curves \( \Gamma_\nu \) on which \( |f(z)| = k_\nu \), by considering \( f(z) \) on \( |z| = R_\nu S_\nu^{1/4} \), and then argue as in Case 2. We assume henceforth that \( m_\nu = 0 \) for all large \( \nu \), so that without loss of generality

\begin{equation}
(3.3) \quad f(z) = 1 + o(1), \quad R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu.
\end{equation}

We also have (2.3), which we write in the form

\begin{equation}
(3.4) \quad f'(z) = D_\nu z^{m_\nu} (1 + \delta(z)), \quad \delta(z) = o(1), \quad R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu,
\end{equation}

and we can assume that \( \delta'(z) = o(1/|z|) \) for the same range of values of \( z \), because otherwise we can replace \( S_\nu \) by \( S_\nu^{1/2} \) and apply Cauchy's estimate for derivatives.
Now if \( n_\nu \leq -2 \) in (3.4), then integration by parts gives, with \( q_\nu = n_\nu + 1 \), 
\( E_\nu = D_\nu q_\nu^{-1}, \ z_0 = R_\nu S_\nu \), and \( L_\nu \) a constant, the estimates

\[
(3.5) \quad f(z) = L_\nu + E_\nu z^{q_\nu}(1 + \delta(z)) - \int_{z_0}^{z} E_\nu t^{q_\nu} \delta'(t) \, dt = L_\nu + E_\nu z^{q_\nu}(1 + o(1)).
\]

In obtaining the last estimate of (3.5) we have taken the path of integration to be the straight line segment from \( z_0 \) to \( |z| \), followed by part of the circle \( |t| = |z| \).

If \( n_\nu = -1 \) in (3.4), then the integral of \( f'(z) \) around the circle \( |z| = R_\nu \) will be \( D_\nu(2\pi i + o(1)) \), which is clearly impossible. Finally if \( n_\nu \geq 0 \) in (3.4), we take \( z_0 = R_\nu S_\nu^{-1} \) and obtain (3.5) again.

Again we consider cases.

**Case A.** Suppose that \( |1 - L_\nu| \geq (1/4)|E_\nu|R_\nu^{q_\nu} \) for infinitely many \( \nu \).

In this case, since \( q_\nu \neq 0 \) we find, using (3.3), that

\[
f(z) - 1 = (L_\nu - 1)(1 + o(1)) = o(1)
\]
either on \( R_\nu S_\nu^{-1} \leq |z| \leq R_\nu S_\nu^{-1/2} \) or on \( R_\nu S_\nu^{1/2} \leq |z| \leq R_\nu S_\nu \), and we can apply the reasoning of Case 1 to \( g(z) = 1/(f(z) - 1) \).

**Case B.** Suppose that \( |1 - L_\nu| < (1/4)|E_\nu|R_\nu^{q_\nu} \) for all large \( \nu \).

Again, since \( q_\nu \neq 0 \), we can obtain, on a smaller annulus formed as in Case A, the estimate \( f(z) - 1 = E_\nu z^{q_\nu}(1 + o(1)) \), and on these annuli \( E_\nu z^{q_\nu} \to 0 \) uniformly, by (3.3). Thus we may apply the reasoning of Case 2 to \( g(z) = 1/(f(z) - 1) \).

4. Proof of Theorem 2

Let \( f_1 \) and \( f_2 \) be linearly independent rational solutions of (1.1), so that the Wronskian \( W(f_1, f_2) = W \) is also rational. Now \( (f_2/f_1)' = W' f_1^{-2} = dz^{q_1-1}(1 + o(1)) \) as \( z \to \infty \), for some nonzero constant \( d \) and integer \( q \), and \( q \) cannot be zero, since \( f_2/f_1 \) is by assumption rational. Therefore we may assume that \( f_2(z)/f_1(z) = z^q(1 + o(1)) \) as \( z \to \infty \) and that \( q \) is positive.

Assuming that \( f \) and \( F \) are as in the statement of Theorem 2, and that \( N(r, 1/fF) = o(T(r, f'/f)) \) and \( f'/f \) is transcendental, we set

\[
(4.1) \quad H(z) = K_1(z)/K_2(z), \quad K_j(z) = f_j(z) - f_j(z)f'(z)/f(z),
\]

so that \( H \) is transcendental of finite lower order.

Now all but finitely many poles of \( H \) are zeros of \( K_2 \) which are not zeros or poles of \( f \). Further, \( K_j(z) = -f_j(z)F(z)/f(z) - K_j(z)(a_1(z) + f'(z)/f(z)) \), so that at a zero \( z \) of \( K_2 \) with \( z \) large and with multiplicity \( m \geq 2 \), \( F(z) \) must have a zero of multiplicity \( m - 1 \). Thus \( N(r, H) - N(r, H) \leq N(r, 1/F) + O(\log r) = o(T(r, H)), \) using (4.1). Moreover,

\[
H'(z) = -W(z)F(z)/f(z)K_2(z)^2,
\]

so that zeros \( z \) of \( H' \) with \( z \) large can only occur at zeros of \( F \) or at simple zeros of \( f \), which implies that \( N(r, 1/H') = o(T(r, H)) \). We may therefore apply Theorem A to \( g(z) \), where \( g(z) \) is either \( H(z) \) or \( 1/(b - H(z)) \), for some constant \( b \), \( g \) being normalized so that \( \delta(\infty, g) = 0 \).

We take a small positive constant \( \epsilon \) and a sequence \( (r_k) \) such that \( r_0 \) is large and \( 2r_k \leq r_{k+1} \leq 4r_k \) for each \( k \geq 0 \) and such that the circles \( |z| = r_k \) do not
meet the exceptional set $C_0$ of Theorem A and further such that $T(r_k, f'/f) \leq O(T(r_k, f))$ for each $k$. Now $L_2(r) = L_2(r_k) + o(1)$, uniformly for $r_k \leq r \leq r_{k+1}$. For each integer $k \geq 0$ we choose $\theta_k^*$ in $[L_2(r_k) - \pi/16 \rho, L_2(r_k) + \pi/16 \rho]$ such that the straight line segments $z = r \exp(i(\theta_k^* + j\pi/\rho))$, $r_k \leq r \leq r_{k+1}$, $0 \leq j \leq 2\rho - 1$, do not meet $C_0$. Obviously, $|\cos(\rho(\theta_k^* - L_2(r_k)))| \geq 3/4$. For each integer $j$ with $0 \leq j \leq 2\rho - 1$ we then choose $\Gamma_j$ to be the union of the straight line segments $z = r \exp(i(\theta_k^* + j\pi/\rho))$, $r_k \leq r \leq r_{k+1}$, and the arcs $z = r_k \exp(i\theta)$, $|\theta - L_2(r_k) - j\pi/\rho| \leq \pi/2\rho - \epsilon$. On $\Gamma_j$, Theorem A gives $|g'(z)| \leq |z|^{-3N}$, where $N$ is a large positive integer, so $|g(z) - A_j| = O(|z|^{-2N})$, for some constant $A_j$. In fact a much stronger estimate is proved in [5], but this suffices for our purposes here and gives either $|H(z) - B_j| = O(|z|^{-2N})$, for some constant $B_j$, or $1/H(z) = O(|z|^{-2N})$. Thus either $G(z) = H(z) f_2(z)/f_1(z)$ or $G(z) = O(1/|z|)$ on $\Gamma_j$, and in either case we obtain there $f'(z)/f(z) = O(1/|z|)$, so $\log^+ |f'(z)/f(z)| = O(\log |z|)$. Now Lemma B implies that for some small constant $\delta$, which satisfies $\delta = O(\epsilon \log(1/\epsilon))$, we have, for each $k \geq 0$,

$$T(r_k, f) \leq (1 + o(1))m(r_k, 1/f) \leq (\delta/2)T(2r_k, 1/f) + O(\log r_k) \leq \delta T(r_{k+1}, f),$$

so, for some positive constant $c$, independent of $\delta$, and for $r_k \leq r \leq r_{k+1}$,

$$T(r, f) \geq T(r_k, f) \geq \delta^{-k} T(r_0, f) \geq \delta^{-c \log r} T(r_0, f) \geq \delta^{-c \log r} T(r_0, f),$$

which contradicts the assumption that $f$ has finite lower order and proves Theorem 2.

**References**


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