PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

UNIQUENESS AND NONUNIQUENESS OF THE POSITIVE CAUCHY PROBLEM FOR THE HEAT EQUATION ON RIEMANNIAN MANIFOLDS

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(Communicated by Peter Li)

ABSTRACT. We investigate a uniqueness problem of whether a nonnegative solution of the heat equation on a noncompact Riemannian manifold is uniquely determined by its initial data. A sufficient condition for the uniqueness (resp. nonuniqueness) is given in terms of nonintegrability (resp. integrability) at infinity of -1 times a negative function by which the Ricci (resp. sectional) curvature of the manifold is bounded from below (resp. above) at infinity. For a class of manifolds, these sufficient conditions yield a simple criterion for the uniqueness.

1. INTRODUCTION

Widder [W] established in 1944 that a nonnegative solution of the heat equation on R^n is determined uniquely by its initial data. This uniqueness theorem was subsequently extended to parabolic equations on Riemannian manifolds (cf. [Ar], [Az], [AT], [Cha], [D2], [Dod], [Don], [Fri], [KT], [KL], [LY], [MT], [M3,4,5], [N], [Su]). Among others, Karp-Li-Yau ([KL] and [LY]) showed that if Ricci curvatures on a geodesic ball of radius R in a complete Riemannian manifold M are bounded from below by $-C_1R^2 - C_2$, where C_1, C_2 are positive constants independent of R, then there holds the Widder uniqueness theorem to the heat equation on M. On the other hand, Azencott [Az] implicitly showed in his study of conservation of probability that if M is a simply connected analytic complete Riemannian manifold whose sectional curvatures on a geodesic sphere of radius R are bounded from above by $-CR^{2+\epsilon}$ for some positive constants C and ϵ, then the Widder uniqueness theorem to the heat equation on M does not hold.

In this paper we give a necessary and sufficient condition for the Widder uniqueness theorem to the heat equation on a noncompact Riemannian manifold to hold.

Let M be an n-dimensional ($n \geq 2$), connected, C^∞, noncompact, complete Riemannian manifold without boundary. Consider a nonnegative classical

Received by the editors July 2, 1993 and, in revised form, September 13, 1993.
1991 *Mathematics Subject Classification.* Primary 58G11, 35K05, 35K15; Secondary 58G32.
Key words and phrases. Uniqueness, positive Cauchy problem, heat equation, Riemannian manifold.

0002-9939/95 $1.00 + $.25 per page

solution of the Cauchy problem

$$(1.1) \qquad (\partial_t - \Delta)u(x, t) = 0 \qquad \text{in} \quad M \times (0, T],$$

$$(1.2) \qquad u(x, 0) = u_0(x) \qquad \text{on} \quad M,$$

where Δ is the Laplace-Beltrami operator for M, $\partial_t = \partial/\partial t$, and T is a positive constant. We say that UPH (uniqueness of the positive Cauchy problem for the heat equation) holds for M when any two nonnegative solutions u and $\tilde{u}$ of the heat equation (1.1) having the same initial value are identically equal on $M \times [0, T]$; note that no global conditions are imposed on solutions.

The purpose of this paper is to point out how curvatures of a noncompact Riemannian manifold determine whether UPH holds for it or not.

In order to state our main results, Theorems A, B, and C below, we need some more notation. Let T_pM be the tangent space to M at a point p in M, and $U_pM = \{\xi \in T_pM; |\xi| = 1\}$. Let $B_p(R)$ be a geodesic ball of radius R centered at p in M : $B_p(R) = \{q \in M; d(p, q) < R\}$. Denote by $K(X, Y)$ the sectional curvature of a plane spanned by linearly independent tangent vectors X and Y, and by $Ric(X)$ the Ricci curvature in the direction X.

Theorem A. *Suppose that $K(R)$ is a positive continuous increasing function on $[0, \infty)$ such that for any $R > 0$*

$$(1.3) \qquad \inf\{Ric(\xi); \ \xi \in U_qM, q \in B_p(R)\} \geq -K(R).$$

If

$$(1.4) \qquad \int_1^\infty \frac{dr}{\max(\sqrt{K(r)}, r)} = \infty,$$

then UPH holds for M.

In the following theorem we shall consider a solution f of the initial value problem

$$(1.5) \qquad f'' = k(r)f \qquad \text{in} \quad (0, \infty),$$

$$(1.6) \qquad f(0) = 0, \quad f'(0) = 1,$$

where k is a continuous function on $[0, \infty)$.

Theorem B. *Assume that the exponential map at a point p is a diffeomorphism from T_pM onto M. Suppose that $k(R)$ is a continuous increasing function on $[0, \infty)$ such that for any $R > 0$*

(1.7)

$$-k(R) \geq \sup\{K(X, \dot{\gamma}(R; q)); \ X \in U_qM, q \in \partial B_p(R), < X, \dot{\gamma}(R; q) >= 0\},$$

where $\gamma(t; q)$ stands for the shortest normal geodesic joining p and q. Assume that $k(R_0) > 0$ for some $R_0 \geq 0$ and the solution f of (1.5)–(1.6) satisfies

$$(1.8) \qquad f > 0 \ \text{in} \ (0, \infty), \qquad \lim_{t \to \infty} f(t) = \infty.$$

If

$$(1.9) \qquad \int_{R_0}^\infty \frac{dr}{\sqrt{k(r)}} < \infty,$$

then UPH does not hold for M.

The following theorem, which is a direct consequence of Theorems A and B, gives a necessary and sufficient condition for UPH to hold for M belonging to some class of Riemannian manifolds.

Theorem C. *Suppose that M is simply connected. Assume that there exist a point p in M, positive constants a, b, and a positive continuous increasing function k on $[0, \infty)$ such that for any $R > 0$, $q \in \partial B_p(R)$, and $X, Y \in U_qM$ with $\langle X, Y\rangle = 0$, the sectional curvature $K(X, Y)$ satisfies*

$$-bk(R) \le K(X, Y) \le -k(R) \le -aR^2. \tag{1.10}$$

Then UPH holds for M if and only if

$$\int_1^\infty \frac{dr}{\sqrt{k(r)}} = \infty. \tag{1.11}$$

The rest of this paper is organized as follows. Theorems A and B are proved in Sections 2 and 3, respectively. In Section 4 we shall apply Theorems A and B to a rotationary symmetric Riemannian manifold M, and show (see Theorem 4.1) that UPH holds for M if and only if -1 times the radial curvature of M satisfies (1.11). In constructing examples the theorem there yields a useful method. In Section 5 we shall give concluding remarks.

The author thanks Yoshiaki Maeda and Hisao Inoue for helpful conversations.

2. Proof of Theorem A

For the proof of Theorem A we make use of a neat uniqueness theorem of Grigor'yan extending that of Täcklind for the heat equation on R^n (cf. [T]).

Lemma 2.1 ([G, Theorem 2]). *Let u be a solution of (1.1)–(1.2) with $u_0 \equiv 0$. Assume that for any $R > 1$*

$$\int_0^T dt \int_{B_p(R)} u(x, t)^2 dV(x) \le Ce^{R\rho(R)}, \tag{2.1}$$

where $\rho(R)$ is an increasing continuous positive function such that

$$\int_1^\infty \frac{dr}{\rho(r)} = \infty. \tag{2.2}$$

Then $u \equiv 0$ on $M \times [0, T]$.

Proof of Theorem A. Let $p(x, y, t)$ be a minimal fundamental solution to the heat equation on M (cf. [Cha]). Then

$$v(x, t) = \int_M p(x, y, t) u_0(y) dV(y) \tag{2.3}$$

is a solution of (1.1)–(1.2) satisfying $0 \le v \le u$ on $M \times [0, T]$. Thus it suffices to show that a nonnegative solution u of (1.1)–(1.2) with $u_0 = 0$ must be identically zero on $M \times [0, T]$. Suppose that u is a nonnegative solution with zero initial data. We see that a function $U(x, t)$ on $M \times [-1, T]$ defined by

$$U(x, t) = \begin{cases} 0 & \text{on } M \times [-1, 0], \\ u(x, t) & \text{on } M \times [0, T] \end{cases}$$

is a nonnegative solution of (1.1) with $(0, T]$ replaced by $(-1, T]$. Thus the proof of Theorems 1.2 and 2.1 of [LY] shows (let the constant α there be 4) that there exists a positive constant C_1 such that

$$u(x, t) \leq u(p, t+s)\left(\frac{t+s+1}{t+1}\right)^{2n}$$

$$\times \exp\left\{\frac{d(p, x)^2}{s} + C_1 s[K(2R) + \frac{1}{R^2} + \sqrt{K(2R)}]\right\}, \tag{2.4}$$

$$x \in B_p(R), \quad R > 1, \quad 0 \leq t < t+s \leq T.$$

Put $J(R) = \max(K(R), R^2)$. Since $\sup_{0<\tau<T} u(p, \tau) < \infty$, there exist positive constants C and m such that

(2.5)

$$u(x, t) \leq C \exp\left(\frac{R^2}{s} + m^2 J(2R)s\right), \quad x \in B_p(R), \quad R > 1, \quad 0 \leq t < t+s \leq T.$$

Fix δ such that $0 < \delta < T$. We claim that

(2.6)

$$u(x, t) \leq C \exp\left([2m + \frac{1}{\delta}]R\sqrt{J(2R)}\right), \quad x \in B_p(R), \quad R > 1, \quad 0 \leq t \leq T - \delta.$$

When $\delta \geq R/m\sqrt{J(2R)}$, we get (2.6) by putting $s = R/m\sqrt{J(2R)}$ in (2.5). When $\delta < R/m\sqrt{J(2R)}$, we have

$$\frac{R^2}{\delta} + m^2 J(2R)\delta \leq \frac{R^2}{\delta} + mR\sqrt{J(2R)} \leq \left(\frac{1}{\delta} + m\right) R\sqrt{J(2R)}.$$

Thus we get (2.6) by putting $s = \delta$ in (2.5). This proves the claim. On the other hand, Bishop's volume comparison theorem (cf. [Sa] and [Cha]) yields

$$vol(B_p(R)) \leq \omega_{n-1} \int_0^R \left(\sqrt{\frac{n-1}{K(R)}} \sinh \sqrt{\frac{K(R)}{n-1}} r\right)^{n-1} dr \tag{2.7}$$

$$\leq C_2 \exp(\sqrt{n-1} R\sqrt{K(2R)})$$

for any $R > 1$, where ω_{n-1} is the area of the $(n-1)$-dimensional unit sphere and C_2 is a positive constant independent of R. Combining (2.6) and (2.7), we obtain

$$\int_0^{T-\delta} dt \int_{B_p(R)} u(x, t)^2 \, dV(x)$$

$$\leq C^2 C_2 (T - \delta) \exp\left([4m + \frac{2}{\delta} + \sqrt{n-1}]R\sqrt{J(2R)}\right)$$

for any $R > 1$. This together with (1.4) and Lemma 2.1 shows that $u = 0$ on $M \times [0, T - \delta]$; which proves Theorem A, since δ can be arbitary small. Q.E.D.

3. Proof of Theorem B

In this section we prove Theorem B almost along the line given in [M5]. In proving it we make use of a comparison theorem due to Bishop, Rauch, etc.

(cf. [Cha], [Ka], [Sa]), and exploit a method developed in connection with nonconservation of probability (cf. [Az], [D1], and [Kh]). The proof is divided into several lemmas; among which Lemmas 3.1 and 3.2 below play a technically crucial role.

Let $k(r)$ be the function given in Theorem B, and $f(r)$ the solution of (1.5)–(1.6). Put $F = f'/f$.

Lemma 3.1. *There exists a positive constant* $R > R_0$ *such that*

$$\inf_{r>R} F(r) > 0, \tag{3.1}$$

$$\int_R^\infty \frac{dr}{F} < \infty, \tag{3.2}$$

$$\inf_{r>R} \frac{k(r)}{F(r)^2} > 0. \tag{3.3}$$

Proof. We have from (1.5)

$$F' + F^2 = k \quad \text{in} \quad (0, \infty). \tag{3.4}$$

By (1.8), there exists $R > R_0$ such that $f(R) > 0$ and $f'(R) > 0$. Let g be a solution of the initial value problem:

$$g'' = k(R)g \quad \text{in} \quad (R, \infty); \qquad g(R) = f(R), \quad g'(R) = f'(R).$$

With $G = g'/g$,

$$(F-G)' + (F+G)(F-G) = k - k(R) \geq 0 \quad \text{in} \quad (R, \infty), \qquad (F-G)(R) = 0.$$

Thus $F \geq G$ on $[R, \infty)$. Since $\lim_{r\to\infty} G(r) = \sqrt{k(R)}$, this implies (3.1). We next claim that

$$\frac{1}{F} + \frac{1}{2}\left(\frac{1}{F^2}\right)' \leq \frac{2}{\sqrt{k}} \qquad \text{in} \quad (R, \infty). \tag{3.5}$$

By (3.4),

$$\frac{1}{k}\frac{F'}{F^2} + \frac{1}{k} = \frac{1}{F^2}.$$

When $F' \geq 0$, $F \leq \sqrt{k}$; and so

$$\frac{1}{F} = F\left(\frac{1}{k} + \frac{1}{k}\frac{F'}{F^2}\right) \leq \frac{1}{\sqrt{k}} + \frac{F'}{F^3}.$$

When $F' < 0$, $1/F \leq 1/\sqrt{k}$; and so

$$\frac{1}{F} + \frac{1}{2}\left(\frac{1}{F^2}\right)' = \frac{1}{F} - \frac{F'}{F^3} = \frac{2}{F} - \frac{k}{F^3} < \frac{2}{\sqrt{k}}.$$

This proves the claim. Integrating (3.5) from R to S we have

$$\int_R^S \frac{dr}{F} + \frac{1}{2}\left(\frac{1}{F(S)^2} - \frac{1}{F(R)^2}\right) \leq \int_R^S \frac{2dr}{\sqrt{k}}.$$

This together with (3.1) and (1.9) implies (3.2). It remains to prove (3.3). Let $H = F/\sqrt{k}$. Since k is increasing, we have for any $r > s \geq R$,

$$F(r) - F(s) \geq \sqrt{k(r)}[H(r) - H(s)].$$

By (3.4),

$$F(r) - F(s) = \int_s^r k(t)[1 - H(t)^2]dt.$$

Thus

$$H(r) - H(s) \le \int_s^r \sqrt{k(t)}[1 - H(t)^2]dt. \tag{3.6}$$

We claim that

$$\sup_{r \ge R} H(r) \le M \equiv \max(1, H(R)). \tag{3.7}$$

On the contrary, suppose that there exists $r > R$ such that $H(r) > M$. Then (3.6) implies

$$\inf\{s \in [R, r]; H(t) \ge H(r) \text{ for any } t \text{ in } [s, r]\} = R.$$

Thus $H(R) \ge H(r) > H(R)$, which is a contradiction. This proves the claim (3.7), which shows (3.3). Q.E.D.

Consider the initial value problem

$$\varphi'' + (n-1)F\varphi' = \varphi \quad \text{in} \quad (R, \infty), \tag{3.8}$$

$$\varphi(R) = 1, \quad \varphi'(R) = 0. \tag{3.9}$$

Lemma 3.2. *$\varphi, \varphi' > 0$ in (R, ∞), and $\varphi(\infty) \equiv \lim_{r\to\infty} \varphi(r) < \infty$.*

Proof. The first assertion clearly holds. Put $\Phi = \log \varphi$. By (3.8),

$$\Phi'' + (n-1)F\Phi' + (\Phi')^2 = 1 \quad \text{in} \quad (R, \infty). \tag{3.10}$$

This implies

$$(n-1)\Phi' \le \frac{1}{F} - \frac{\Phi''}{F}.$$

We have

$$-\int_R^r \frac{\Phi''}{F} ds = \frac{\Phi'(R)}{F(R)} - \frac{\Phi'(r)}{F(r)} + \int_R^r \Phi' \left(\frac{1}{F}\right)' ds \le \frac{\Phi'(R)}{F(R)} + \int_R^r \Phi' \left(1 - \frac{k}{F^2}\right) ds$$

Thus

$$\int_R^r \Phi' \left(n - 2 + \frac{k}{F^2}\right) ds \le \int_R^r \frac{ds}{F} + \frac{\Phi'(R)}{F(R)}. \tag{3.11}$$

Hence

$$\left(\inf_{r>R} \frac{k(r)}{F(r)^2}\right) (\Phi(r) - \Phi(R)) \le \int_R^\infty \frac{ds}{F} + \frac{\Phi'(R)}{F(R)}$$

for any $r > R$; which shows that $\lim_{r\to\infty} \varphi(\infty) < \infty$. Q.E.D.

Put

$$\psi(r) = 1 - \frac{\varphi(r)}{\varphi(\infty)}, \qquad \Psi(x) = \psi(d(p, x)).$$

Lemma 3.3. *ψ is decreasing,* $0 < \psi < 1$, $\lim_{r\to\infty} \psi(r) = 0$, *and*

$$(1 - \Delta)\Psi \geq 1 \quad \textit{in} \quad M \setminus \bar{B}_p(R). \tag{3.12}$$

Proof. We have only to prove (3.12). By geodesic spherical coordinates based at p,

$$-\Delta\Psi(x) = \frac{1}{\varphi(\infty)} \left(\varphi''(r) + \frac{\partial_r\sqrt{\mathbf{g}}}{\sqrt{\mathbf{g}}} \varphi'(r) \right),$$

where $r = d(p, x)$ and $\sqrt{\mathbf{g}}$ is a density function of the area element of $\partial B_p(r)$ with respect to the standard area element of the unit sphere S^{n-1} (cf. [Cha, p. 149]). We now apply a comparison theorem (cf. [Ka, Theorem 2.49] and [Sa, Theorem 3.1]) to $\partial_r\sqrt{\mathbf{g}}/\sqrt{\mathbf{g}}$, taking as a model manifold a rotationary symmetric Riemannian manifold with radial sectional curvature $-k(r)$ (see [Cho] or Section 4 below), and get

$$\frac{\partial_r\sqrt{\mathbf{g}}}{\sqrt{\mathbf{g}}} \geq (n-1)F,$$

where $F = f'/f$ with $f(r)$ being the solution of (1.5)–(1.6). Since $\varphi' > 0$, this implies that

$$(1 - \Delta)\Psi(x) \geq 1 + \frac{1}{\varphi(\infty)} \left(\varphi''(r) + (n-1)F(r)\varphi'(r) - \varphi(r) \right) = 1.$$

Q.E.D.

We are now ready to complete the proof of Theorem B by constructing a positive null solution. Let $p(x, y, t)$ be a minimal fundamental solution for the heat equation on M. Put

$$w(x, t) = \int_M p(x, y, t)\, dV(y). \tag{3.13}$$

Then we see that w is a solution of (1.1) with $w(x, 0) = 1$ and $0 < w \leq 1$ on $M \times [0, \infty)$.

Lemma 3.4. $0 < w < 1$ *in* $M \times (0, \infty)$.

Proof. Put

$$v(x) = \int_0^\infty e^{-t} w(x, t)\, dt. \tag{3.14}$$

Then $0 < v \leq 1$ and $(1 - \Delta)v = 1$ on M. By Lemma 3.3 and the minimality of $p(x, y, t)$, there exists a positive constant C such that

$$v(x) \leq C\Psi(x) \quad \text{in} \quad M \setminus \bar{B}_p(R) \tag{3.15}$$

(cf. [D1, Lemma 2.3]). Thus

$$\lim_{d(p,x)\to\infty} v(x) = 0, \tag{3.16}$$

which implies that $w \not\equiv 1$. On the other hand, the parabolic Harnack inequality together with the semigroup property of the minimal fundamental solution p implies that either $w \equiv 1$ or $0 < w < 1$ in $M \times (0, \infty)$. Hence $0 < w < 1$. Q.E.D.

Completion of the proof of Theorem B. Put $u(x, t) = 1 - w(x, t)$. Then we see that u is a solution of (1.1) with $u(x, 0) = 0$ and $0 < u < 1$ in $M \times (0, \infty)$. Q.E.D.

4. Rotationary symmetric Riemannian manifolds

Let M be a Riemannian manifold rotationary symmetric at p such that the exponential map at p is a diffeomorphism from T_pM onto M (cf. [Cho]). Then the sectional curvature $K(X, \dot{\gamma}(R; q))$ in (1.7) depends only on R, and is called a radial curvature. Denote it by $-k(R)$, and let f be a solution of (1.5)–(1.6). Then the Riemannian metric in terms of a geodesic polar coordinates at p is given by

$$ds^2 = dr^2 + f(r)^2 d\Theta^2, \tag{4.1}$$

where $d\Theta^2$ is the standard metric of the unit sphere S^{n-1}.

Theorem 4.1. *Assume that* k *is a positive increasing function on* $[0, \infty)$ *such that* $k(r) \geq ar^2$ *on* $[0, \infty)$, *where* a *is a positive constant. Then UPH holds for* M *if and only if* (1.11) *holds.*

In view of Theorems A and B, Theorem 4.1 is derived from the following lemma.

Lemma 4.2. *There exists a positive constant* C *such that*

$$K(X, Y) \geq -Ck(R) \tag{4.2}$$

for any $R > 0$ *and* $X, Y \in U_qM$ *with* $\langle X, Y\rangle = 0$ *and* $q \in \partial B_p(R)$.

Proof. With $X = (x, v)$, $Y = (0, w)$, where $x \in [-1, 1]$ and $v, w \in T_\theta S^{n-1}$ $(q = (R, \theta))$, the sectional curvature $K(X, Y)$ is given by

$$K(X, Y) = \frac{1 - f'(R)^2}{f(R)^2}(1 - x^2) - \frac{f''(R)}{f(R)}x^2 \tag{4.3}$$

(cf. [BO, the formula on p. 27]). Thus, in view of (1.5), it suffices to estimate the function

$$g(R) \equiv \frac{1 - f'(R)^2}{f(R)^2}.$$

Clearly, $\lim_{R\to 0} g(R) = -k(0)$ and $f, f' > 0$ in $(0, \infty)$. By (3.3) of Lemma 3.1, there exists a positive constant $m > 0$ such that $k(R) \geq m[f'(R)/f(R)]^2$ for any $R > 1$. Hence there exists a positive constant C such that $g(R) \geq -Ck(R)$ for any $R > 0$. This proves the lemma. Q.E.D.

5. Remarks

5.1. When UPH does not hold for M, an interesting problem is to determine the structure of all positive solutions u of (1.1)–(1.2) with $u_0 = 0$ and $u > 0$ in $M \times (0, T]$. This problem is closely related to a parabolic Martin boundary and Martin kernel (cf. [Fre], [MT], [M3,4], [P]).

5.2. It is of some interest to compare Theorem 4.1 with a delicate criterion on existence of a nonconstant positive harmonic function, which is unstable under constant multiplication of the radial curvature (cf. [M2,1]).

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