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ISOMORPHISMS OF STANDARD OPERATOR ALGEBRAS

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ABSTRACT. Let X and Y be Banach spaces, $\dim X = \infty$, and let $\mathscr{A}$ and $\mathscr{B}$ be standard operator algebras on X and Y, respectively. Assume that $\phi: \mathscr{A} \to \mathscr{B}$ is a bijective mapping satisfying $\|\phi(AB)-\phi(A)\phi(B)\| \leq \varepsilon$, $A, B \in \mathscr{A}$, where ε is a given positive real number (no linearity or continuity of ϕ is assumed). Then ϕ is a spatially implemented linear or conjugate linear algebra isomorphism. In particular, ϕ is continuous.

Let X be a Banach space. By $\mathscr{B}(X)$ we mean the algebra of all bounded linear operators on X. We denote by $\mathscr{F}(X)$ the subalgebra of bounded finite rank operators. We shall call a subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$ standard provided $\mathscr{A}$ contains $\mathscr{F}(X)$ ($\mathscr{A}$ need not be closed). For any $x \in X$ and $f \in X'$ we denote by $x \otimes f$ the bounded linear operator on X defined by $(x \otimes f)y = f(y)x$ for $y \in X$. Note that every operator of rank one can be written in this form. The operator $x \otimes f$ is a projection if and only if $f(x) = 1$.

Let X and Y be Banach spaces, and let $\mathscr{A}$ and $\mathscr{B}$ be standard operator algebras on X and Y, respectively. It is a classical result [4] that every algebra isomorphism $\phi: \mathscr{A} \to \mathscr{B}$ is spatial, i.e., there is a linear topological isomorphism $T: X \to Y$ such that $\phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$.

When discussing isomorphisms of algebras one usually assumes that these mappings are linear. A more general approach is to consider the algebra only as a ring. It seems that the first step in this direction was made by Rickart [9, Theorem 3.2], who treated isomorphisms of primitive real Banach algebras which are not assumed to be linear, i.e., they are isomorphisms merely in the ring sense. The famous result of Kaplansky [6, 7] decomposes a ring isomorphism between two semisimple complex Banach algebras into a linear part, a conjugate linear part, and a nonreal linear part on a finite-dimensional ideal.

Let R be a ring. Recall that R is called prime if $aRb = 0$ implies $a = 0$ or $b = 0$. Assume that a prime ring R contains an idempotent $e \neq 0, 1$ (R need not have an identity). Then every multiplicative bijective mapping of R onto an arbitrary ring S is additive [8]. It is an easy consequence of the Hahn-Banach theorem that $\mathscr{B}(X)$ is a prime ring. Thus, the above-mentioned results imply that if $\dim X = \infty$, then every multiplicative bijective mapping ϕ of $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ is of the form $\phi(A) = TAT^{-1}$, where $T: X \to Y$ is

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either a linear topological isomorphism or a conjugate linear topological isomorphism. We shall see later that the assumption that X is infinite dimensional is indispensable in this statement. Let us mention here that a similar result concerning bijective multiplicative mappings between real Banach algebras was obtained by Rickart [9, Lemma 3.1].

Instead of homomorphisms one can study approximate homomorphisms. For example, Bourgin [3] proved that if ϕ is a map of a normed algebra $\mathscr{A}$ onto a normed algebra $\mathscr{B}$, and ϕ satisfies both $\|\phi(x+y)-\phi(x)-\phi(y)\|<\varepsilon$ and $\|\phi(xy)-\phi(x)\phi(y)\|<\delta$, $x, y \in \mathscr{A}$, for given positive real numbers ε, δ, then ϕ is a ring homomorphism of $\mathscr{A}$ onto $\mathscr{B}$. In the same paper, he considered transformations $\phi: C(S_1) \to C(S_2)$ which satisfy $\|\phi(fg)-\phi(f)\phi(g)\|<\varepsilon$ for all $f, g \in C(S_1)$. Here $C(S_i)$, $i=1,2$, denotes the algebra of continuous functions over the compact space S_i. Under some rather mild conditions, he proved that such a mapping ϕ is actually a multiplicative transformation. In [2] Baker showed that an unbounded approximately multiplicative complex-valued function defined on a semigroup S is multiplicative. An interested reader can find further references on problems concerning approximate homomorphisms in a survey paper [5].

The above-mentioned results lead naturally to the following question: Is every approximately multiplicative bijective mapping $\phi: \mathscr{B}(X) \to \mathscr{B}(Y)$ a spatially implemented linear or conjugate linear algebra isomorphism? Moreover, is the same true for approximately multiplicative bijective mappings between arbitrary standard operator algebras? It is the aim of this note to show that the answer to this question is the affirmative in the case that $\dim X = \infty$. We shall also discuss the finite-dimensional case. It should be mentioned that one of the ideas that we will use is similar to those of Baker [2], who proved the superstability of multiplicative complex-valued functions.

Theorem. *Let X and Y be Banach spaces, $\dim X = \infty$, and let $\mathscr{A}$ and $\mathscr{B}$ be standard operator algebras on X and Y, respectively. Let $\varepsilon > 0$, and assume that $\phi: \mathscr{A} \to \mathscr{B}$ is a bijective mapping satisfying $\|\phi(AB)-\phi(A)\phi(B)\| \le \varepsilon$ for all $A, B \in \mathscr{A}$. Then ϕ is of the form $\phi(A) = TAT^{-1}$, $A \in \mathscr{A}$, where $T: X \to Y$ is either a bounded linear bijective operator or a bounded conjugate linear bijective operator.*

Proof. First we shall prove that the mapping ϕ is actually multiplicative. Let C be a bounded linear finite rank operator on Y. Since ϕ is surjective, there exists a bounded linear operator $D \in \mathscr{A}$ such that $\phi(D) = C$. For arbitrary $A, B \in \mathscr{A}$ we have

$$\begin{aligned}\|(\phi(AB)-\phi(A)\phi(B))C\| &= \|(\phi(AB)-\phi(A)\phi(B))\phi(D)\| \\ &\le \|\phi(AB)\phi(D)-\phi(A)\phi(BD)\| + \|\phi(A)\phi(BD)-\phi(A)\phi(B)\phi(D)\| \\ &\le \|\phi(AB)\phi(D)-\phi(ABD)\| + \|\phi(ABD)-\phi(A)\phi(BD)\| \\ &\quad + \|\phi(A)\|\,\|\phi(BD)-\phi(B)\phi(D)\| \\ &\le \varepsilon(2+\|\phi(A)\|).\end{aligned}$$

Replacing C by nC and sending n to infinity we see that $\phi(AB)-\phi(A)\phi(B)$ annihilates $\mathscr{F}(Y)$, and, therefore, ϕ is multiplicative.

The standard operator algebra $\mathscr{A}$ is a prime ring. It follows that ϕ is a ring isomorphism [8, Corollary].

Let us fix a vector $z \in X$ and a bounded linear functional $g \in X'$ satisfying $g(z) = 1$. The operator $z \otimes g$ is a projection of rank one. Consequently, $\phi(z \otimes g)$ is a projection, and we claim that it has rank one, too. Indeed, if this is not so, we have $\phi(z \otimes g) = Q_1 + Q_2$, where Q_i, $i = 1, 2$, are nonzero projections and $\operatorname{rank} Q_1 = 1$. It follows that Q_1 belongs to $\mathscr{F}(Y) \subset \mathscr{B}$. The same must be true for $Q_2 = \phi(z \otimes g) - Q_1$. As ϕ is a ring isomorphism we can find two nonzero projections $P_1, P_2 \in \mathscr{A}$ such that $\phi(P_i) = Q_i$, $i = 1, 2$. This further implies that $z \otimes g = P_1 + P_2$—a contradiction. Thus, we have $\phi(z \otimes g) = u \otimes h$ for some $u \in Y$ and $h \in Y'$ satisfying $h(u) = 1$.

We define an additive mapping $T\colon X \to Y$ by

$$Tx = \phi(x \otimes g)u, \qquad x \in X.$$

Let A be any operator from $\mathscr{A}$. For an arbitrary $x \in X$ we have

$$TAx = \phi(A(x \otimes g))u = \phi(A)\phi(x \otimes g)u = \phi(A)Tx.$$

Hence, $TA = \phi(A)T$ holds for every $A \in \mathscr{A}$.

Our next step will be to prove that T is bijective. Let us first assume that we have $Tx = 0$ for some nonzero $x \in X$. We choose f from X' such that $f(x) = 1$. It follows that

$$0 = \phi(z \otimes f)Tx = \phi((z \otimes f)(x \otimes g))u = u,$$

which contradicts the fact that $h(u) = 1$.

Let us choose a vector $w \in Y$. In order to prove that T is surjective we must show that $w \in \operatorname{Im} T$. For this purpose we fix a nonzero vector $x \in X$. As $Tx \neq 0$ we can find $k \in Y'$ such that $k(Tx) = 1$. The surjectivity of ϕ yields the existence of an operator $A \in \mathscr{A}$ satisfying $\phi(A) = w \otimes k$. It is easy to see that the equation $TAx = \phi(A)Tx$ implies the desired relation $w \in \operatorname{Im} T$.

Next, we shall see that $\dim Y > 1$. Otherwise we would have $\mathscr{B} \approx \mathbb{C}$. This would imply that $\mathscr{A}$ and $\mathbb{C}$ are isomorphic rings, which contradicts the fact that $\mathscr{F}(X)$ possesses nontrivial zero divisors.

Let x be an arbitrary nonzero vector from X. We choose $f \in X'$ such that $f(x) = 1$, so that $x \otimes f$ is a projection. We already know that $\phi(x \otimes f)$ can be written as $y \otimes m$ for some $y \in Y$ and $m \in Y'$. Applying $TA = \phi(A)T$ with $A = x \otimes f$ we get for an arbitrary nonzero complex number λ that

$$\begin{aligned} 0 \neq T(\lambda x) &= T(x \otimes f)(\lambda x) \\ &= \phi(x \otimes f)T(\lambda x) = m(T(\lambda x))y. \end{aligned}$$

Hence, we have $T(\lambda x) \in \operatorname{span}\{Tx\}$ for every $x \in X$ and every complex number λ. This yields that for every nonzero $x \in X$ there exists an additive function $\tau_x\colon \mathbb{C} \to \mathbb{C}$ such that

$$T(\lambda x) = \tau_x(\lambda)Tx$$

holds for all $\lambda \in \mathbb{C}$. Let x and y be vectors from X such that Tx and Ty are linearly independent. Comparing $T(\lambda(x + y))$ with $T(\lambda x) + T(\lambda y)$ one can see that $\tau_x = \tau_{x+y} = \tau_y$. If Tx and Ty are nonzero linearly dependent vectors, we have $\tau_x = \tau_z = \tau_y$, where z from X was chosen in such a way that Tx and Tz are linearly independent. It follows that τ_x is independent of x, and consequently, there exists an additive function $\tau\colon \mathbb{C} \to \mathbb{C}$ such that

$$T(\lambda x) = \tau(\lambda)Tx, \qquad x \in X, \lambda \in \mathbb{C}.$$

Now we shall prove that τ is a ring homomorphism of $\mathbb{C}$. Indeed,

$$\tau(\lambda\mu)Tx = T(\lambda\mu x) = \tau(\lambda)T(\mu x) = \tau(\lambda)\tau(\mu)Tx.$$

It is easy to see that the range of τ contains the field of all rational numbers and that $T^{-1}(py) = pT^{-1}y$ holds for all $p \in \mathbb{Q}$ and all $y \in Y$.

It should be mentioned that so far we have not used the assumption that X is infinite dimensional. We shall need it for the proof of the continuity of the function τ. Let us assume on the contrary that τ is not continuous. Then τ is unbounded on every neighborhood of 0. We will construct by induction sequences $(y_n) \subset Y$ and $(f_n) \subset X'$ satisfying

$$\|y_i\| < 2^{-i} \quad \text{and} \quad \|f_i\| < 2^{-i}$$

for all positive integers i,

$$f_i(T^{-1}y_k) = 0$$

for all positive $i \neq k$, and

$$|\tau(f_n(T^{-1}y_n))| > n + \sum_{i=1}^{n-1} |\tau(f_i(T^{-1}y_i))|$$

for all integers $n > 1$. We choose a vector $y_1 \in Y$ and a functional $f_1 \in X'$ such that $\|y_1\| < 2^{-1}$ and $\|f_1\| < 2^{-1}$. Suppose that we have already found vectors $y_1, \dots, y_n$ and functionals $f_1, \dots, f_n$ having the above properties. We denote a closed complementary subspace of $\operatorname{span}\{T^{-1}y_1, \dots, T^{-1}y_n\}$ in X by V_n and choose a nonzero vector $y_{n+1} \in Y$ such that $T^{-1}y_{n+1} \in (\bigcap_{i=1}^n \operatorname{Ker} f_i) \cap V_n = Z_n$. Observe that Z_n is nontrivial because it is an intersection of two subspaces of finite codimension in an infinite-dimensional space. With no loss of generality we may assume that the norm of y_{n+1} is smaller than 2^{-n-1}, otherwise we can multiply it by a small enough positive rational number. We can find a functional $g_{n+1} \in X'$, $\|g_{n+1}\| < 2^{-n-1}$, such that g_{n+1} annihilates the set $\{T^{-1}y_1, \dots, T^{-1}y_n\}$ while $g_{n+1}(T^{-1}y_{n+1}) \neq 0$. As the set $\{\tau(\lambda g_{n+1}(T^{-1}y_{n+1})) : \lambda \in \mathbb{C},\ |\lambda| < 1\}$ is unbounded, we can find a λ such that $f_{n+1} = \lambda g_{n+1}$ has the desired properties.

Let us introduce now a bounded linear functional $f = \sum_{i=1}^\infty f_i$ and a sequence of vectors $w_n = \sum_{i=1}^n y_i$, $n \in \mathbb{N}$. Let $x \in X$ be any nonzero vector. The operator $\phi(x \otimes f) = T(x \otimes f)T^{-1}$ is bounded, but on the other hand we have that

$$\begin{aligned}\|\phi(x\otimes f)w_n\| &= \left\| T(x\otimes f)T^{-1}\left(\sum_{i=1}^n y_i\right)\right\| = \left\|\tau\left(f\left(\sum_{i=1}^n T^{-1}y_i\right)\right)Tx\right\| \\ &= \left|\tau\left(\sum_{i=1}^n f_i(T^{-1}y_i)\right)\right| \|Tx\| > n\|Tx\|\end{aligned}$$

which contradicts the fact that the sequence (w_n) is bounded. Thus, τ is continuous, and consequently it must be either of the form $\tau(\lambda) = \lambda$ or of the form $\tau(\lambda) = \bar{\lambda}$ [1, pp. 52–57].

It follows that T is either linear or conjugate linear bijective mapping. Using the closed-graph theroem we shall show that it is also continuous. Obviously, ϕ maps the set of all linear bounded rank-one operators on X onto the set of all linear bounded rank-one operators on Y. Moreover, for every rank-one operator $A \in \mathscr{F}(X)$ the operator TA is bounded. The same must be true for the operator $\phi(A)T$. Thus, for every $v \in Y$ and $k \in Y'$ the operator $(v \otimes k)T$ is continuous. Let $(x_n) \subset X$ be a sequence satisfying $x_n \to x$ and $Tx_n \to y$. It follows that

$$\begin{aligned} k(y)v = (v \otimes k)y &= \lim_{n\to\infty}((v \otimes k)T)x_n \\ &= (v \otimes k)(Tx) = k(Tx)v, \end{aligned}$$

and consequently, $Tx = y$. This completes the proof.

We shall conclude by considering the case that X is finite dimensional. Then $\mathscr{A}$ is isomorphic to the algebra of all $n \times n$ matrices for some positive integer n. Consequently, the center of $\mathscr{A}$ is nonempty, and the same must be true for the center of $\mathscr{B}$. Thus, $\dim Y < \infty$, and ϕ can be considered as a mapping between matrix algebras M_n and M_m. Here, M_k denotes the set of all $k \times k$ matrices. Moreover, ϕ maps the center of M_n onto the center of M_m. This yields together with $\phi(\lambda I_n) = \tau(\lambda)I_m$ that τ is a ring automorphism of $\mathbb{C}$. Introducing a new mapping $\varphi\colon M_n \to M_m$ given by

$$\varphi([a_{ij}]) = \phi([\tau^{-1}(a_{ij})]),$$

one can easily verify that φ is an algebra isomorphism. This further implies $n = m$ and the existence of an invertible $n \times n$ matrix T such that

$$\phi([a_{ij}]) = T[\tau(a_{ij})]T^{-1}$$

for all $[a_{ij}] \in M_n$.

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