\( \sigma \)-WEAKLY CLOSED MODULES
OF CERTAIN REFLEXIVE OPERATOR ALGEBRAS

CHEN PEIXIN

(Communicated by Palle E. T. Jorgensen)

Abstract. Let \( \mathcal{A} \) be a completely distributive CSL algebra and let \( M \) be any \( \sigma \)-weakly closed \( \mathcal{A} \)-module. We give characterizations of commutant \( C(\mathcal{A}, M) \) of \( \mathcal{A} \) modulo \( M \) and \( \text{AlgLat} M \). Furthermore, we deal with the relations among \( \mathcal{A} \), \( C(\mathcal{A}, M) \) and \( \text{AlgLat} M \).

1. Preliminaries and notation

Throughout this paper, \( H \) will denote a complex Hilbert space, and \( L(H) \) will denote the Banach algebra of all bounded linear operators from \( H \) into itself. Let \( \mathcal{S} \) be a commutative and completely distributive subspace lattice on \( H \) and let \( \mathcal{A} = \text{Alg} \mathcal{S} \subseteq L(H) \). A subspace \( \mathcal{M} \) of \( L(H) \) is said to be a \( \mathcal{A} \)-module if \( \mathcal{A} \mathcal{M} = \mathcal{M} \mathcal{A} \subseteq \mathcal{M} \). Any \( \sigma \)-weakly closed \( \mathcal{A} \)-module is denoted by \( M \) and the set of all \( V \)-generators of \( \mathcal{A} \) is denoted by \( \mathcal{G} \). The terminology and notation of this paper concerning reflexive operator algebras may be found in [1, 3, 4]. It is known from [3] that \( M \) has the form

\[
M = \{ T \in L(H) : TE \subseteq \tilde{E} \text{ for all } E \in \mathcal{S} \}
\]

where \( E \rightarrow \tilde{E} \) is some lattice homomorphism of \( \mathcal{S} \) into itself. Thus we need only consider \( M \) determined by the homomorphism \( E \rightarrow \tilde{E} \).

2. Commutants relative to \( \sigma \)-weakly closed \( \mathcal{A} \)-modules

In [1] the authors found that \( C(\text{Alg} \mathcal{N}, M) = \mathcal{G}_M \oplus M \) for a nest \( \mathcal{N} \), where \( \mathcal{G}_M \) is a weakly closed subspace (is also a subalgebra) of the core of \( \text{Alg} \mathcal{N} \). A natural question is: Does this hold for \( \mathcal{S} \) ? In this case when \( M \supseteq \mathcal{A} \) Han Deguang proved \( C(\mathcal{A}, M) = M \) [3]. Generally, we obtain \( C(\mathcal{A}, M) = \mathcal{G}_M(G) \oplus M \), where \( \mathcal{G}_M(G) \) is a weakly closed subspace of the core of \( \mathcal{A} \).

We omit the proof of Lemma 2.1 as it follows by modification of the arguments in [4, p. 505].

Received by the editors June 25, 1993 and, in revised form, September 23, 1993.
1991 Mathematics Subject Classification. Primary 47D25, 47D15; Secondary 47B47.
Key words and phrases. \( \mathcal{A} \)-module, reflexive operator algebra, commutant of \( \mathcal{A} \) modulo module.
Lemma 2.1. If $T \in \mathcal{A}'$, then each $V$-generator of $\mathcal{L}$ is contained in an eigenspace corresponding to the eigenvalue $\lambda$ of $T$ and $\|\lambda\| \leq \|T\|$. 

Lemma 2.2. If $E \in \mathcal{L}$ and $T \in C(\mathcal{A}, M)$, then 

(i) $(I - \tilde{E})TEG = \lambda_{TE}(G)(E - \tilde{EE})G$ and $|\lambda_{TE}(G)| \leq \|T\|$, where $G$ is any $V$-generator of $\mathcal{L}$ and $\lambda_{TE}(G)$ is a number depending on $T$, $E$, and $G$; 

(ii) if $\tilde{EE} \leq \tilde{F}F < E < F$ 

$$\lambda_{TE}(G) = \lambda_{TF}(G),$$

where $F \in \mathcal{L}$, and $\lambda_{TE}(G)$ and $\lambda_{TF}(G)$ are as in (i). 

Proof. (i) Since $E \in \mathcal{A}$, $I - \tilde{E} \in \mathcal{A}$, and $T \in C(\mathcal{A}, M)$ 

$$(I - \tilde{E})(TE - ET)E = 0, \quad (I - \tilde{E})[T(I - \tilde{E}) - (I - \tilde{E})T]E = 0.$$ 

Therefore 

$$(I - \tilde{E})T(E - \tilde{EE}) = (I - \tilde{E})TE = (E - \tilde{EE})TE$$ 

and hence 

$$(I - \tilde{E})TE = (E - \tilde{EE})T(E - \tilde{EE}).$$ 

Now for any $A \in \mathcal{A}$ 

$$(I - \tilde{E})(TA - AT)E = 0$$ 

and since $(E - \tilde{EE})A(E - \tilde{EE}) \in \mathcal{A}$ 

$$(I - \tilde{E})TE[(E - \tilde{EE})A(E - \tilde{EE})] - [(E - \tilde{EE})A(E - \tilde{EE})](I - \tilde{E})TE$$ 

$$= (I - \tilde{E})[T(E - \tilde{EE})A(E - \tilde{EE}) - (E - \tilde{EE})A(E - \tilde{EE})T]E = 0.$$ 

This means that $(I - \tilde{E})TE = (E - \tilde{EE})T(E - \tilde{EE}) \in [(E - \tilde{EE})\mathcal{A} (E - \tilde{EE})]'$. 

Apply Lemma 2.1 to the compression of $\mathcal{A}$ to the range of $E - \tilde{EE}$ 

$$(I - \tilde{E})TEG = [(E - \tilde{EE})T(E - \tilde{EE})][((E - \tilde{EE})G(E - \tilde{EE})]$$ 

$$= \lambda_{TE}(G)[(E - \tilde{EE})G(E - \tilde{EE})]$$ 

$$= \lambda_{TE}(G)G,$$ 

$$|\lambda_{TE}(G)| \leq \|(E - \tilde{EE})T(E - \tilde{EE})\| \leq \|T\|.$$ 

(ii) Observe 

$$(E - \tilde{EE})T(E - \tilde{EE})G = \lambda_{TE}(G)(E - \tilde{EE})G,$$ 

$$(F - \tilde{FF})T(F - \tilde{FF})G = \lambda_{TF}(G)(F - \tilde{FF})G.$$ 

Multiplying the above on either side by $E - \tilde{FF}$, we obtain 

$$(E - \tilde{FF})T(E - \tilde{FF})G = \lambda_{TE}(G)(E - \tilde{FF})G,$$ 

$$(E - \tilde{FF})T(E - \tilde{FF})G = \lambda_{TF}(G)(E - \tilde{FF})G.$$ 

Thus 

$$\lambda_{TE}(G) = \lambda_{TF}(G),$$ 

and (ii) follows. 

Remark 2.3. Lemma 2.2 shows that for $T \in C(\mathcal{A}, M)$ and $G \in \mathcal{F}$, $\lambda_{TE}(G)$ has a constant value on the intervals $(\tilde{E}_1E_1, E_2]$ and $(\tilde{E}_2E_2, E_1]$ if intervals $(\tilde{E}_1E_1, E_1)$ and $(\tilde{E}_2E_2, E_2)$ overlap. It follows easily that $\lambda_{TE}(G)$ is constant on each maximal connected component of $\bigcup\{(\tilde{EE}, E]: E \in \mathcal{L}\}$. 

Definition. The element $F$ of $\mathcal{L}$ is \(~\) connected to $E$ (notation $E \lessgtr F$ or $F \gtrless E$) if $E = F$ or $E < F$ and exists a finite chain $E_n < \cdots < E_1 < E_0 = F$ with $E_i E_{i+1} < E_{i+1}$ ($0 \leq i \leq n-1$) and $E_n E < E$.

Let $\mathcal{L}_0 = \{E \in \mathcal{L}: \tilde{E} E < E\}$. For each $E \in \mathcal{L}_0$ we define the \(~\)-component $\gamma(E)$ of $E$ by

$$\gamma(E) = \{F \in \mathcal{L}: F \lessgtr E\} \cup \{F \in \mathcal{L}: F \gtrless E\}.$$ 

Clearly, $\mathcal{L}_0$ is a disjoint union of \(~\)-components and it is easy to see that \(~\)-components are intervals. We may write

$$\mathcal{L}_0 = \bigcup_{\omega \in \Omega} \gamma_\omega$$

where $\Omega$ is some index set and $\{\gamma_\omega: \omega \in \Omega\}$ are pairwise disjoint intervals with left end point $E_\omega$ and right end point $F_\omega$.

For fixed $T \in C(\mathcal{M}, M)$ and $F \in \mathcal{L}$, it is possible that $G_1, G_2 \in \mathcal{M}$ and $G_1 \neq G_2$ but $\lambda_T F(G_1) = \lambda_T F(G_2)$. Let $G_T F(\alpha)$ be the closed linear span of all $v$-generators of $\mathcal{L}$ corresponding to the same eigenvalue $\lambda_T F(\alpha)$ of $(I - \tilde{F})TF$, where $\alpha \in \Lambda$ and $\Lambda$ is some index set. Lemma 2.2 shows that

$$G_T E(\alpha) = G_T F(\alpha)$$

for all $E, F \in \gamma_\omega$; we denote it by $G_T(\omega)(\alpha)$. Clearly, $G_T(\omega)(\alpha) \in \mathcal{L}$ and $I = \bigvee_{\alpha \in \Lambda} G_T(\omega)(\alpha)$. If $T \in C(\mathcal{M}, M), \omega \in \Omega$ are given and $E \in \gamma_\omega$, then

$$(E - \tilde{E})G_T(\omega)(\alpha)G_T(\omega)(\beta) = G_T(\omega)(\alpha)G_T(\omega)(\beta)(E - \tilde{E})$$

$$= \begin{cases} (E - \tilde{E})G_T(\omega)(\alpha) & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Denote by $\mathcal{M}_M(G)$ the weakly closed linear span generated by the projections $\{(F_\omega - E_\omega)G_T(\omega)(\alpha): T \in C(\mathcal{M}, M), \omega \in \Omega, \text{ and } \alpha \in \Lambda\}$. 

Theorem 2.4. $C(\mathcal{M}, M) = \mathcal{M}_M(G) \oplus M$, where the sum is a direct sum of vector spaces.

Proof. Suppose $T \in C(\mathcal{M}, M), \omega \in \Omega, \text{ and } \alpha \in \Lambda$ are given. For any $E \in \mathcal{L}$, there is a \(~\)-component $\gamma_{\omega_0}$ of $\mathcal{L}_0$ such that $E \in \gamma_{\omega_0}$. For any $A \in \mathcal{M}$, if $\omega_0 \neq \omega$,

$$(I - \tilde{E})[A(F_\omega - E_\omega)G_T(\omega)(\alpha) - (F_\omega - E_\omega)G_T(\omega)(\alpha)A]E = 0;$$

if $\omega_0 = \omega$, then for any $\beta \in \Lambda$

$$(E - \tilde{E})[AG_T(\omega)(\alpha) - G_T(\omega)(\alpha)A](E - \tilde{E})G_T(\omega)(\beta)$$

$$= (E - \tilde{E})G_T(\omega)(\alpha)G_T(\omega)(\beta)A(E - \tilde{E})G_T(\omega)(\alpha)G_T(\omega)(\beta)$$

$$- (E - \tilde{E})G_T(\omega)(\alpha)G_T(\omega)(\beta)A(E - \tilde{E})G_T(\omega)(\beta)$$

$$= 0.$$
Hence
\[(I - \tilde{E})[A(F - E)G_{T\omega}(\alpha) - (F - E)G_{T\omega}(\alpha)A]E = (I - \tilde{E})A[(I - \tilde{E})(F - E)E]G_{T\omega}(\alpha) - [(I - \tilde{E})(F - E)E]G_{T\omega}(\alpha)AE = (E - \tilde{E})[AG_{T\omega}(\alpha) - G_{T\omega}(\alpha)A](E - \tilde{E})E = (E - \tilde{E})[AG_{T\omega}(\alpha) - G_{T\omega}(\alpha)A](E - \tilde{E}) \left( \bigvee_{\beta \in \Lambda} G_{T\omega}(\beta) \right) = 0,\]
so for all \( S \in \mathcal{M}(G) \)
\[(I - \tilde{E})(AS - SA)E = 0.\]
This shows that \( S \in C(\mathcal{A}, M) \), and thus the inclusion \( \mathcal{M}(G) + M \subseteq C(\mathcal{A}, M) \) follows.

Now suppose \( T \in C(\mathcal{A}, M) \); then it follows from Lemma 2.2 that if \( \gamma_\omega \) is any \( \sim \)-component of \( \mathcal{L}_0 \) and \( E \in \gamma_\omega \), we have
\[\lambda_{T\omega}(E)G_{T\omega}(\alpha) = \lambda_{T\omega}(\alpha)(E - \tilde{E})E_{T\omega}(\alpha)\]
where \( \lambda_{T\omega}(\alpha) \) is a number depending on \( T, \gamma_\omega \), and \( \alpha \). Define \( T_M \) by
\[T_M = \sum_{\omega \in \Omega, \alpha \in \Lambda} \lambda_{T\omega}(\alpha)(F - F)G_{T\omega}(\alpha).\]

By Lemma 2.2, \( |\lambda_{T\omega}(\alpha)| \leq ||T|| \); hence, the series converges in the strong operator topology and \( T_M \in \mathcal{M}(G) \).

For any \( \beta \in \Lambda \)
\[(I - \tilde{E})(T - T_M)E_{T\omega}(\beta) = (I - \tilde{E})T\omega(\beta) - (I - \tilde{E})T_M E_{T\omega}(\beta) = \lambda_{T\omega}(\beta)(E - \tilde{E})G_{T\omega}(\beta) - \lambda_{T\omega}(\beta)(E - \tilde{E})G_{T\omega}(\beta) = 0.\]

Hence
\[(I - \tilde{E})(T - T_M)E = (I - \tilde{E})(T - T_M)E \left( \bigvee_{\beta \in \Lambda} G_{T\omega}(\beta) \right) = 0\]
and \( T - T_M \in M \). Therefore \( T \in \mathcal{M}(G) + M \).

To prove that the sum is direct, if \( T \in \mathcal{M}(G) \), then for each \( \omega \in \Omega \) and any \( \beta \in \Lambda \)
\[T(F - E_\omega)G_{T\omega}(\beta) = \lambda_{T\omega}(\beta)(F - E_\omega)G_{T\omega}(\beta).\]
If also \( T \in M \), choose \( E \in \gamma_\omega \)
\[O = (I - \tilde{E})T\omega(\beta) = \lambda_{T\omega}(\beta)(E - \tilde{E})G_{T\omega}(\beta).\]
Thus \( \lambda_{T\omega}(\beta) = 0 \) for each \( \omega \in \Omega \) and each \( \beta \in \Lambda \), and
\[T(F - E_\omega) = T(F - E_\omega) \left( \bigvee_{\beta \in \Lambda} G_{T\omega}(\beta) \right) = 0.\]
Therefore \( T = 0 \). The proof is completed.
3. The operator algebra generated by a module

In this section, first we prove that $\text{AlgLat} M = \mathcal{C}_M \oplus \langle M \rangle$, where $\langle M \rangle$ is the weakly closed algebra generated by $M$. Furthermore, we deal with the relations among $\text{AlgLat} M$, $C(\mathcal{A}, \langle M \rangle)$, and $\mathcal{A}$.

Lemma 3.1 is the analogue of Lemma 1.10 in [1], thus we omit the proof.

Lemma 3.1. The weakly closed algebra generated by $M$ is the module determined by $E \rightarrow \widetilde{E}$, where

$$
\begin{align*}
\widetilde{E} &= \begin{cases}
\tilde{E} & \text{if } \tilde{E} \leq E, \\
\bigvee \{E^{(n)} : n \geq 0\} & \text{if } \tilde{E} \not\leq E.
\end{cases}
\end{align*}
$$

Lemma 3.2 [3]. $P \in \text{Lat} M$ if and only if there exists some $E \in \mathcal{L}$ such that $\widetilde{E} \leq P \leq E$.

Theorem 3.3. $\text{AlgLat} M = \mathcal{C}_M \oplus \langle M \rangle$, where $\mathcal{C}_M$ is the weakly closed algebra generated by $\{F_\omega - E_\omega : \omega \in \Omega_0\}$. $F_\omega$ and $E_\omega$ are the end points of the ~-components of $\{E \in \mathcal{L} : \tilde{E} < E\}$, $\Omega_0 \subseteq \Omega$, and $\langle M \rangle$ is the weakly closed algebra generated by $M$.

Proof. Clearly $\text{AlgLat} M = \text{AlgLat} \langle M \rangle$. Since $\mathcal{C}_M$ depends only on the elements $E$ of $\mathcal{L}$ such that $\tilde{E} < E$, Lemma 3.1 shows that $\mathcal{C}_M = \mathcal{C}_M$. Thus we need only prove the equation

$$
\text{AlgLat} \langle M \rangle = \mathcal{C}_M \oplus \langle M \rangle.
$$

That $\text{AlgLat} \langle M \rangle \supseteq \mathcal{C}_M + \langle M \rangle$ is obvious. Let $T \in \text{AlgLat} \langle M \rangle$. From Lemma 3.2 for all $E \in \mathcal{L}$ with $\tilde{E} \leq E$ and any $G \leq E - \tilde{E}$, $\tilde{E} + G \in \text{Lat} T$. Thus $(E - \tilde{E})T(E - \tilde{E})$ leaves every subprojection of $E - \tilde{E}$ invariant. This means that

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_{TE}(E - \tilde{E})$$

for some scalar $\lambda_{TE}$. Note that $E \in \text{Lat} \langle M \rangle \subseteq \text{Lat} T$; we have

$$(E - \tilde{E})TE = (E - \tilde{E})T(E - \tilde{E}) = \lambda_{TE}(E - \tilde{E}).$$

If $E$ and $F$ are in the same ~-component it follows as in Lemma 2.2 that $\lambda_{TE} = \lambda_{TF}$. The proof is now completed by modification of the arguments in Theorem 2.4.

Definition. $M$ is said to have property $(\ast)$ if for any $T \in C(\mathcal{A}, M)$, $E \in \mathcal{L}$, there is a number $\lambda_{TE}$ such that

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_{TE}(E - \tilde{E}).$$

The following facts are easily seen.

(i) $M \supseteq \mathcal{A}$ iff $\tilde{E} \geq E$ for all $E \in \mathcal{L}$;
(ii) $M \subseteq \mathcal{A}$ iff $\tilde{E} \leq E$ for all $E \in \mathcal{L}$;
(iii) $\mathcal{C}_M(G) \subseteq \mathcal{A}$.

We can prove the following results by the above facts and Theorems 2.4 and 3.3.

Corollary 3.4. Let $\langle M \rangle$ be the weakly closed algebra generated by $M$. Then

(i) $\text{AlgLat} M \subseteq C(\mathcal{A}, \langle M \rangle)$;
(ii) if $M \supseteq \mathcal{A}$, then $\text{AlgLat} M = \langle M \rangle$ and $C(\mathcal{A}, M) = M$;
(iii) if $M \subseteq \mathcal{A}$, then $\text{AlgLat} M \subseteq C(\mathcal{A}, M) \subseteq \mathcal{A}$.
Furthermore, AlgLat $M = C(\mathcal{A}, M)$ iff $M$ has property ($*$); $C(\mathcal{A}, M) = \mathcal{A}$ iff $(E - \bar{E})(A_1 A_2 - A_2 A_1)(E - \bar{E}) = 0$ for all $A_1, A_2 \in \mathcal{A}$ and $E \in \mathcal{L}$.

ACKNOWLEDGMENT

I am grateful to vice professor Han Deguang for his help and the referee for many suggestions.

REFERENCES


DEPARTMENT OF MATHEMATICS AND PHYSICS, UNIVERSITY OF PETROLEUM, DONGYING CITY, SHANDONG PROVINCE, 257062, PEOPLES REPUBLIC OF CHINA