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σ-WEAKLY CLOSED MODULES OF CERTAIN REFLEXIVE OPERATOR ALGEBRAS

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ABSTRACT. Let $\mathscr{A}$ be a completely distributive CSL algebra and let M be any σ-weakly closed $\mathscr{A}$-module. We give characterizations of commutant $C(\mathscr{A}, M)$ of $\mathscr{A}$ modulo M and $\operatorname{AlgLat} M$. Furthermore, we deal with the relations among $\mathscr{A}$, $C(\mathscr{A}, M)$ and $\operatorname{AlgLat} M$.

1. PRELIMINARIES AND NOTATION

Throughout this paper, H will denote a complex Hilbert space, and $L(H)$ will denote the Banach algebra of all bounded linear operators from H into itself. Let $\mathscr{L}$ be a commutative and completely distributive subspace lattice on H and let $\mathscr{A} = \operatorname{Alg}\mathscr{L} \subseteq L(H)$. A subspace $\mathscr{M}$ of $L(H)$ is said to be a $\mathscr{A}$-module if $\mathscr{A}\mathscr{M} = \mathscr{M}\mathscr{A} \subseteq \mathscr{M}$. Any σ-weakly closed $\mathscr{A}$-module is denoted by M and the set of all V-generators of $\mathscr{L}$ is denoted by $\mathscr{G}$. The terminology and notation of this paper concerning reflexive operator algebras may be found in [1, 3, 4]. It is known from [3] that M has the form

$$M = \{T \in L(H) \colon TE \subseteq \widetilde{E} \text{ for all } E \in \mathscr{L}\}$$

where $E| \to \widetilde{E}$ is some lattice homomorphism of $\mathscr{L}$ into itself. Thus we need only consider M determined by the homomorphism $E| \to \widetilde{E}$.

2. COMMUTANTS RELATIVE TO σ-WEAKLY CLOSED $\mathscr{A}$-MODULES

In [1] the authors found that $C(\operatorname{Alg}\mathscr{N}, M) = \mathscr{C}_M \oplus M$ for a nest $\mathscr{N}$, where $\mathscr{C}_M$ is a weakly closed subspace (is also a subalgebra) of the core of $\operatorname{Alg}\mathscr{N}$. A natural question is: Does this hold for $\mathscr{L}$? In this case when $M \supseteq \mathscr{A}$ Han Deguang proved $C(\mathscr{A}, M) = M$ [3]. Generally, we obtain $C(\mathscr{A}, M) = \mathscr{C}_M(G) \oplus M$, where $\mathscr{C}_M(G)$ is a weakly closed subspace of the core of $\mathscr{A}$.

We omit the proof of Lemma 2.1 as it follows by modification of the arguments in [4, p. 505].

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Lemma 2.1. *If $T \in \mathscr{A}'$, then each $\vee$-generator of $\mathscr{L}$ is contained in an eigenspace corresponding to the eigenvalue λ of T and $\|\lambda\| \leq \|T\|$.*

Lemma 2.2. *If $E \in \mathscr{L}$ and $T \in C(\mathscr{A}, M)$, then*

(i) $(I - \widetilde{E})TEG = \lambda_{TE}(G)(E - \widetilde{E}E)G$ *and* $|\lambda_{TE}(G)| \leq \|T\|$, *where G is any V-generator of $\mathscr{L}$ and $\lambda_{TE}(G)$ is a number depending on T, E, and G;*

(ii) *if* $\widetilde{E}E \leq \widetilde{F}F < E < F$

$$\lambda_{TE}(G) = \lambda_{TF}(G),$$

where $F \in \mathscr{L}$, and $\lambda_{TE}(G)$ and $\lambda_{TF}(G)$ are as in (i).

Proof. (i) Since $E \in \mathscr{A}$, $I - \widetilde{E} \in \mathscr{A}$, and $T \in C(\mathscr{A}, M)$

$$(I - \widetilde{E})(TE - ET)E = 0, \qquad (I - \widetilde{E})[T(I - \widetilde{E}) - (I - \widetilde{E})T]E = 0.$$

Therefore

$$(I - \widetilde{E})T(E - \widetilde{E}E) = (I - \widetilde{E})TE = (E - \widetilde{E}E)TE$$

and hence

$$(I - \widetilde{E})TE = (E - \widetilde{E}E)T(E - \widetilde{E}E).$$

Now for any $A \in \mathscr{A}$

$$(I - \widetilde{E})(TA - AT)E = 0$$

and since $(E - \widetilde{E}E)A(E - \widetilde{E}E) \in \mathscr{A}$

$$\begin{aligned}(I - \widetilde{E})TE[(E - \widetilde{E}E)A(E - \widetilde{E}E)] &- [(E - \widetilde{E}E)A(E - \widetilde{E}E)](I - \widetilde{E})TE \\ &= (I - \widetilde{E})[T(E - \widetilde{E}E)A(E - \widetilde{E}E) - (E - \widetilde{E}E)A(E - \widetilde{E}E)T]E = 0.\end{aligned}$$

This means that $(I - \widetilde{E})TE = (E - \widetilde{E}E)T(E - \widetilde{E}E) \in [(E - \widetilde{E}E)\mathscr{A}(E - \widetilde{E}E)]'$.

Apply Lemma 2.1 to the compression of $\mathscr{A}$ to the range of $E - \widetilde{E}E$

$$\begin{aligned}(I - \widetilde{E})TEG &= [(E - \widetilde{E}E)T(E - \widetilde{E}E)][(E - \widetilde{E}E)G(E - \widetilde{E}E)] \\ &= \lambda_{TE}(G)[(E - \widetilde{E}E)G(E - \widetilde{E}E)] \\ &= \lambda_{TE}(G)(E - \widetilde{E}E)G,\end{aligned}$$

$$|\lambda_{TE}(G)| \leq \|(E - \widetilde{E}E)T(E - \widetilde{E}E)\| \leq \|T\|.$$

(ii) Observe

$$\begin{aligned}(E - \widetilde{E}E)T(E - \widetilde{E}E)G &= \lambda_{TE}(G)(E - \widetilde{E}E)G, \\ (F - \widetilde{F}F)T(F - \widetilde{F}F)G &= \lambda_{TF}(G)(F - \widetilde{F}F)G.\end{aligned}$$

Multiplying the above on either side by $E - \widetilde{F}F$, we obtain

$$\begin{aligned}(E - \widetilde{F}F)T(E - \widetilde{F}F)G &= \lambda_{TE}(G)(E - \widetilde{F}F)G, \\ (E - \widetilde{F}F)T(E - \widetilde{F}F)G &= \lambda_{TF}(G)(E - \widetilde{F}F)G.\end{aligned}$$

Thus

$$\lambda_{TE}(G) = \lambda_{TF}(G),$$

and (ii) follows.

Remark 2.3. Lemma 2.2 shows that for $T \in C(\mathscr{A}, M)$ and $G \in \mathscr{G}$, $\lambda_{TE}(G)$ has a constant value on the intervals $(\widetilde{E}_1E_1, E_2]$ and $(\widetilde{E}_2E_2, E_1]$ if intervals $(\widetilde{E}_1E_1, E_1)$ and $(\widetilde{E}_2E_2, E_2)$ overlap. It follows easily that $\lambda_{TE}(G)$ is constant on each maximal connected component of $\bigcup\{(\widetilde{E}E, E]: E \in \mathscr{L}\}$.

Definition. The element F of $\mathscr{L}$ is $\sim$ connected to E (notation $E \lesssim F$ or $F \gtrsim E$) if $E = F$ or $E < F$ and exists a finite chain $E_n < \cdots < E_1 < E_0 = F$ with $\widetilde{E}_i E_i < E_{i+1}$ $(0 \le i \le n-1)$ and $\widetilde{E}_n E < E$.

Let $\mathscr{L}_0 = \{E \in \mathscr{L} : \widetilde{E}E < E\}$. For each $E \in \mathscr{L}_0$ we define the $\sim$-component $\gamma(E)$ of E by

$$\gamma(E) = \{F \in \mathscr{L} : F \lesssim E\} \cup \{F \in \mathscr{L} : F \gtrsim E\}.$$

Clearly, $\mathscr{L}_0$ is a disjoint union of $\sim$-components and it is easy to see that $\sim$-components are intervals. We may write

$$\mathscr{L}_0 = \bigcup_{\omega \in \Omega} \gamma_\omega$$

where Ω is some index set and $\{\gamma_\omega : \omega \in \Omega\}$ are pairwise disjoint intervals with left end point E_ω and right end point F_ω.

For fixed $T \in C(\mathscr{A}, M)$ and $F \in \mathscr{L}$, it is possible that G_1, $G_2 \in \mathscr{G}$ and $G_1 \neq G_2$ but $\lambda_{TF}(G_1) = \lambda_{TF}(G_2)$. Let $G_{TF}(\alpha)$ be the closed linear span of all $\vee$-generators of $\mathscr{L}$ corresponding to the same eigenvalue $\lambda_{TF}(\alpha)$ of $(I - \widetilde{F})TF$, where $\alpha \in \Lambda$ and Λ is some index set. Lemma 2.2 shows that

$$G_{TE}(\alpha) = G_{TF}(\alpha)$$

for all E, $F \in \gamma_\omega$; we denote it by $G_{T\omega}(\alpha)$. Clearly, $G_{T\omega}(\alpha) \in \mathscr{L}$ and $I = \bigvee_{\alpha \in A} G_{T\omega}(\alpha)$. If $T \in C(\mathscr{A}, M)$, $\omega \in \Omega$ are given and $E \in \gamma_\omega$, then

$$\begin{aligned}(E - \widetilde{E}E)G_{T\omega}(\alpha)G_{T\omega}(\beta) &= G_{T\omega}(\alpha)G_{T\omega}(\beta)(E - \widetilde{E}E)\\ &= \begin{cases} (E - \widetilde{E}E)G_{T\omega}(\alpha) & \text{if } \alpha = \beta,\\ 0 & \text{if } \alpha \neq \beta.\end{cases}\end{aligned}$$

Denote by $\mathscr{C}_M(G)$ the weakly closed linear span generated by the projections $\{(F_\omega - E_\omega)G_{T\omega}(\alpha) : T \in C(\mathscr{A}, M), \omega \in \Omega, \text{ and } \alpha \in \Lambda\}$.

Theorem 2.4. *$C(\mathscr{A}, M) = \mathscr{C}_M(G) \oplus M$, where the sum is a direct sum of vector spaces.*

Proof. Suppose $T \in C(\mathscr{A}, M)$, $\omega \in \Omega$, and $\alpha \in \Lambda$ are given. For any $E \in \mathscr{L}$, there is a $\sim$-component γ_{ω_0} of $\mathscr{L}_0$ such that $E \in \gamma_{\omega_0}$. For any $A \in \mathscr{A}$, if $\omega_0 \neq \omega$,

$$(I - \widetilde{E})[A(F_\omega - E_\omega)G_{T\omega}(\alpha) - (F_\omega - E_\omega)G_{T\omega}(\alpha)A]E = 0;$$

if $\omega_0 = \omega$, then for any $\beta \in \Lambda$

$$\begin{aligned}&(E - \widetilde{E}E)[AG_{T\omega}(\alpha) - G_{T\omega}(\alpha)A](E - \widetilde{E}E)G_{T\omega}(\beta)\\ &\quad = (E - \widetilde{E}E)G_{T\omega}(\alpha)G_{T\omega}(\beta)A(E - \widetilde{E}E)G_{T\omega}(\alpha)G_{T\omega}(\beta)\\ &\qquad - (E - \widetilde{E}E)G_{T\omega}(\alpha)G_{T\omega}(\beta)A(E - \widetilde{E}E)G_{T\omega}(\beta)\\ &\quad = 0.\end{aligned}$$

Hence

$$\begin{aligned}
&(I-\widetilde{E})[A(F_\omega - E_\omega)G_{T\omega}(\alpha) - (F_\omega - E_\omega)G_{T\omega}(\alpha)A]E \\
&\quad = (I-\widetilde{E})A[(I-\widetilde{E})(F_\omega - E_\omega)E]G_{T\omega}(\alpha) - [(I-\widetilde{E})(F_\omega - E_\omega)E]G_{T\omega}(\alpha)AE \\
&\quad = (E-\widetilde{E}E)[AG_{T\omega}(\alpha) - G_{T\omega}(\alpha)A](E-\widetilde{E}E) \\
&\quad = (E-\widetilde{E}E)[AG_{T\omega}(\alpha) - G_{T\omega}(\alpha)A](E-\widetilde{E}E)\left(\bigvee_{\beta\in\Lambda} G_{T\omega}(\beta)\right) \\
&\quad = 0,
\end{aligned}$$

so for all $S \in \mathscr{C}_M(G)$

$$(I-\widetilde{E})(AS - SA)E = 0.$$

This shows that $S \in C(\mathscr{A}, M)$, and thus the inclusion $\mathscr{C}_M(G)+M \subseteq C(\mathscr{A}, M)$ follows.

Now suppose $T \in C(\mathscr{A}, M)$; then it follows from Lemma 2.2 that if γ_{ω_0} is any $\sim$-component of $\mathscr{L}_0$ and $E \in \gamma_{\omega_0}$, we have

$$(I-\widetilde{E})TEG_{T\omega_0}(\alpha) = \lambda_{T\omega_0}(\alpha)(E-\widetilde{E}E)G_{T\omega_0}(\alpha)$$

where $\lambda_{T\omega_0}(\alpha)$ is a number depending on T, γ_{ω_0}, and α. Define T_M by

$$T_M = \sum_{\omega\in\Omega}\sum_{\alpha\in A} \lambda_{T\omega}(\alpha)(F_\omega - F_\omega)G_{T\omega}(\alpha).$$

By Lemma 2.2, $|\lambda_{T\omega}(\alpha)| \le \|T\|$; hence, the series converges in the strong operator topology and $T_M \in \mathscr{C}_M(G)$.

For any $\beta \in \Lambda$

$$\begin{aligned}
(I-\widetilde{E})(T-T_M)EG_{T\omega_0}(\beta) &= (I-\widetilde{E})TEG_{T\omega_0}(\beta) - (I-\widetilde{E})T_MEG_{T\omega_0}(\beta) \\
&= \lambda_{T\omega_0}(\beta)(E-\widetilde{E}E)G_{T\omega_0}(\beta) - \lambda_{T\omega_0}(\beta)(E-\widetilde{E}E)G_{T\omega_0}(\beta) \\
&= 0.
\end{aligned}$$

Hence

$$(I-\widetilde{E})(T-T_M)E = (I-\widetilde{E})(T-T_M)E\left(\bigvee_{\beta\in\Lambda} G_{T\omega_0}(\beta)\right) = 0$$

and $T - T_M \in M$. Therefore $T \in \mathscr{C}_M(G) + M$.

To prove that the sum is direct, if $T \in \mathscr{C}_M(G)$, then for each $\omega \in \Omega$ and any $\beta \in \Lambda$

$$T(F_\omega - E_\omega)G_{T\omega}(\beta) = \lambda_{T\omega}(\beta)(F_\omega - E_\omega)G_{T\omega}(\beta).$$

If also $T \in M$, choose $E \in \gamma_\omega$

$$O = (I-\widetilde{E})TEG_{T\omega}(\beta) = \lambda_{T\omega}(\beta)(E-\widetilde{E}E)G_{T\omega}(\beta).$$

Thus $\lambda_{T\omega}(\beta) = 0$ for each $\omega \in \Omega$ and each $\beta \in \Lambda$, and

$$T(F_\omega - E_\omega) = T(F_\omega - E_\omega)\left(\bigvee_{\beta\in\Lambda} G_{T\omega}(\beta)\right) = 0.$$

Therefore $T = 0$. The proof is completed.

3. The operator algebra generated by a module

In this section, first we prove that $\operatorname{AlgLat} M = \mathscr{C}_M \oplus \langle M \rangle$, where $\langle M \rangle$ is the weakly closed algebra generated by M. Furthermore, we deal with the relations among $\operatorname{AlgLat} M$, $C(\mathscr{A}, \langle M \rangle)$, and $\mathscr{A}$.

Lemma 3.1 is the analogue of Lemma 1.10 in [1], thus we omit the proof.

Lemma 3.1. *The weakly closed algebra generated by M is the module determined by $E| \to \widehat{E}$, where*

$$\widehat{E} = \begin{cases} \widetilde{E} & \text{if } \widetilde{E} \leq E, \\ \bigvee\{\widetilde{E}^{(n)} : n \geq 0\} & \text{if } \widetilde{E} \nleq E. \end{cases}$$

Lemma 3.2 [3]. *$P \in \operatorname{Lat} M$ if and only if there exists some $E \in \mathscr{L}$ such that $\widetilde{E} \leq P \leq E$.*

Theorem 3.3. $\operatorname{AlgLat} M = \mathscr{C}_M \oplus \langle M \rangle$, *where $\mathscr{C}_M$ is the weakly closed algebra generated by $\{F_\omega - E_\omega : \omega \in \Omega_0\}$. F_ω and E_ω are the end points of the $\sim$-components of $\{E \in \mathscr{L} : \widetilde{E} < E\}$, $\Omega_0 \subseteq \Omega$, and $\langle M \rangle$ is the weakly closed algebra generated by M.*

Proof. Clearly $\operatorname{AlgLat} M = \operatorname{AlgLat}\langle M \rangle$. Since $\mathscr{C}_M$ depends only on the elements E of $\mathscr{L}$ such that $\widetilde{E} < E$, Lemma 3.1 shows that $\mathscr{C}_M = \mathscr{C}_{\langle M \rangle}$. Thus we need only prove the equation

$$\operatorname{AlgLat}\langle M \rangle = \mathscr{C}_{\langle M \rangle} \oplus \langle M \rangle.$$

That $\operatorname{AlgLat}\langle M \rangle \supseteq \mathscr{C}_{\langle M \rangle} + \langle M \rangle$ is obvious. Let $T \in \operatorname{AlgLat}\langle M \rangle$. From Lemma 3.2 for all $E \in \mathscr{L}$ with $\widetilde{E} \leq E$ and any $G \leq E - \widetilde{E}$, $\widetilde{E} \oplus G \in \operatorname{Lat} T$. Thus $(E - \widetilde{E})T(E - \widetilde{E})$ leaves every subprojection of $E - \widetilde{E}$ invariant. This means that

$$(E - \widetilde{E})T(E - \widetilde{E}) = \lambda_{TE}(E - \widetilde{E})$$

for some scalar λ_{TE}. Note that $E \in \operatorname{Lat}\langle M \rangle \subseteq \operatorname{Lat} T$; we have

$$(I - \widetilde{E})TE = (E - \widetilde{E})T(E - \widetilde{E}) = \lambda_{TE}(E - \widetilde{E}).$$

If E and F are in the same $\sim$-component it follows as in Lemma 2.2 that $\lambda_{TE} = \lambda_{TF}$. The proof is now completed by modification of the arguments in Theorem 2.4.

Definition. M is said to have property $(*)$ if for any $T \in C(\mathscr{A}, M)$, $E \in \mathscr{L}$, there is a number λ_{TE} such that

$$(E - \widetilde{E})T(E - \widetilde{E}) = \lambda_{TE}(E - \widetilde{E}).$$

The following facts are easily seen.

(i) $M \supseteq \mathscr{A}$ iff $\widetilde{E} \geq E$ for all $E \in \mathscr{L}$;
(ii) $M \subseteq \mathscr{A}$ iff $\widetilde{E} \leq E$ for all $E \in \mathscr{L}$;
(iii) $\mathscr{C}_M(G) \subseteq \mathscr{A}$.

We can prove the following results by the above facts and Theorems 2.4 and 3.3.

Corollary 3.4. *Let $\langle M \rangle$ be the weakly closed algebra generated by M. Then*

(i) $\operatorname{AlgLat} M \subseteq C(\mathscr{A}, \langle M \rangle)$;
(ii) *if $M \supseteq \mathscr{A}$, then* $\operatorname{AlgLat} M = \langle M \rangle$ *and* $C(\mathscr{A}, M) = M$;
(iii) *if $M \subseteq \mathscr{A}$, then* $\operatorname{AlgLat} M \subseteq C(\mathscr{A}, M) \subseteq \mathscr{A}$.

Furthermore, $\mathrm{AlgLat}\, M = C(\mathscr{A}, M)$ *iff* M *has property* $(*)$; $C(\mathscr{A}, M) = \mathscr{A}$ *iff* $(E - \widetilde{E})(A_1A_2 - A_2A_1)(E - \widetilde{E}) = 0$ *for all* $A_1, A_2 \in \mathscr{A}$ *and* $E \in \mathscr{L}$.

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