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UNIQUENESS OF MAXIMAL ENTROPY ODD ORBIT TYPES

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ABSTRACT. We prove that the maximal entropy orbit types of odd period for interval maps are unique. In fact we prove that they are uniquely maximal among all (not necessarily cyclic) permutations.

1. INTRODUCTION

Periodic orbits of interval maps can be classified combinatorially using the order inherited from the interval. For a continuous map, the existence of a periodic orbit of a given type typically entails the existence of many other orbit types and implies a lower bound on the topological entropy of the map. Several authors have studied the implications among orbit types and the relation between orbit types and entropy, including for example [Be, BCop, BGMY, GT, J] and the systematic treatment of [MN].

If $p_1 < \cdots < p_n$ is any periodic orbit of a (continuous) map f of a compact interval, we define the *type* of this orbit as the cyclic permutation θ on $\{1, \ldots, n\}$ given by $\theta(i) := j$ if $f(p_i) = p_j$, $1 \le i \le n$. More generally, if $S = \{p_1, \ldots, p_n\}$, $p_1 < \cdots < p_n$, and $f(S) = S$, we define the *type* of the finite invariant set S to be the permutation θ given as before by $\theta(i) := j$ if $f(p_i) = p_j$.

We write C_n for the set of all possible orbit types, or cycles, of period n, and P_n for the set of all permutations on $\{1, \ldots, n\}$, and set $C := \bigcup_{n\ge1} C_n$, $P := \bigcup_{n\ge1} P_n$.

The *dual* of a permutation $\theta \in P_n$ is the permutation $\overline{\theta} \in P_n$, $\overline{\theta}(i) = n + 1 - \theta(n + 1 - i)$, so that $\overline{\theta}$ is θ conjugated by a reversal of orientation.

Definition 1 [MN]. The entropy of an orbit type $\theta \in C$ is

$$h(\theta) := \inf\{h(f) : f \text{ is a map with an orbit of type } \theta\}.$$

Here $h(f)$ is the topological entropy of the map f. Since the topological entropy of a map represents its dynamical complexity in an appropriate sense, the entropy of an orbit type θ can be thought of as representing the dynamical complexity required for any map with a periodic orbit of type θ.

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More generally, if $\theta \in P$, we define $h(\theta) := \inf\{h(f) : f$ has an invariant set of type $\theta\}$. It is easy to see that for all $\theta \in P$, $h(\overline{\theta}) = h(\theta)$.

A permutation $\theta \in P$ *forces* $\eta \in P$ if every map having an invariant set of type θ also has one of type η.

In order to avoid handling the trivial case $n = 1$, we establish the convention that " n odd" will mean " $n > 1$ odd" for the remainder of the paper.

For n odd, define $l := \lfloor (n-1)/4 \rfloor$ so that if $n \equiv 1 \pmod 4$, then $n = 4l+1$ and if $n \equiv 3 \pmod 4$, then $n = 4l+3$. Define the orbit type θ_n of period n by

$$j \mapsto \begin{cases} n-2l-j & \text{if } 1 \le j < n-2l \text{ and } j \text{ odd;} \\ j-n+2l+1 & \text{if } n-2l \le j \le n \text{ and } j \text{ odd;} \\ n-2l+j-1 & \text{if } 1 \le j \le 2l \text{ and } j \text{ even;} \\ n+2l-j+2 & \text{if } 2l < j \le n \text{ and } j \text{ even.} \end{cases}$$

It can easily be verified that the θ_n are in fact cycles.

For example, using cycle notation we have $\theta_{11} = (1\ 6\ 11\ 5\ 2\ 8\ 9\ 3\ 4\ 10\ 7)$. Misiurewicz and Nitecki [MN] first considered these orbit types, with $n \equiv 1 \pmod 4$.

It was proved in [GT] that the orbit types θ_n have maximal entropy among all n-cycles, and in fact among all n-permutations:

Theorem 1. *For n odd,* $h(\theta_n) = \max\{h(\theta) : \theta \in P_n\}$.

We extend here the methods of [GT] to prove these orbit types are the *unique* maximal entropy odd types (up to duality).

Theorem 2. *For n odd, if $\eta \in P_n$ and* $h(\eta) = \max\{h(\theta) : \theta \in P_n\}$, *then $\eta = \theta_n$ or $\eta = \overline{\theta}_n$.*

When n is even, the maximal entropy cycles are not known. However, it is known that the maximal entropy permutations are not cycles, and come in self-dual pairs, in contrast to the situation for n odd. A forthcoming paper of the first author and Zhang will address this.

2. Uniqueness

Proof of Theorem 2. For $\theta \in P_n$, define the *primitive function* f_θ on the interval $[1, n]$ as the piecewise linear interpolation of θ. Define $M(\theta)$ to be the $(n-1)$ by $(n-1)$ matrix whose (i, j)th entry is 1 if $f_\theta([i, i+1]) \supset [j, j+1]$ and 0 otherwise. It is well known [BCop] that for $\theta \in P$, $h(\theta) = \log \lambda$ where λ is the spectral radius of $M(\theta)$.

For odd n, let $\eta \in P_n$ have maximal entropy, i.e., $h(\eta) = \max\{h(\theta) : \theta \in P_n\}$. We need to show that $\eta = \theta_n$ or $\eta = \overline{\theta}_n$. Note that by the remarks preceding Theorem 11.6 in [MN], η will not be forced by any other element of P_n and f_η will be maximodal and have all maximum values above all minimum values. Here f_η *maximodal* means that it has a local extremum at $1, 2, \ldots, n \in [1, n]$. Since n is odd and η is maximodal, either η or $\overline{\eta}$ is *normalized*, i.e., either f_η or $f_{\overline{\eta}}$ has a local minimum at $1 \in [1, n]$. Without loss of generality, assume η is normalized. Then we show that $\eta = \theta_n$.

Let $rA = \lambda r$, where $A = M(\theta_n)$ and $h(\theta_n) = \log \lambda$, so that r is a (left) Perron-Frobenius eigenvector for A and λ is the spectral radius of A. Set $B = M(\eta)$ so that λ is also the spectral radius of B.

Let $\varepsilon > 0$. We define the matrix $B^\varepsilon = (B^\varepsilon_{ij})$ by

$$B^\varepsilon_{ij} = B_{ij} + \varepsilon \left[1 - \prod_{i=1}^{n-1} \delta(A_{ij}, B_{ij}) \right],$$

where $\delta(\cdot, \cdot)$ is Kronecker's delta. In other words, $B^\varepsilon_{ij} = B_{ij} + \varepsilon$ if the jth column of B differs from the jth column of A, and otherwise $B^\varepsilon_{ij} = B_{ij}$.

Then B^ε is irreducible.

This is true because B^ε has a positive entry wherever A has a positive entry, i.e., if $A_{ij} > 0$, then $B^\varepsilon_{ij} > 0$, since either $B^\varepsilon_{ij} = B_{ij} = A_{ij} > 0$ or $B^\varepsilon_{ij} = B_{ij} + \varepsilon \geq \varepsilon > 0$. But we will see in the remark following Proposition 1 that A is irreducible; therefore B^ε is also irreducible.

Let $k = (n-1)/2$. Consider the Euclidean space $R^{n-1} = R^{2k}$. A vector w in this space will be denoted for notational convenience as

$$w = (u, v) = (u_1, \dots, u_k, v_k, \dots, v_1).$$

At this point we need to introduce a cone, first defined in [GT], which will enable us to obtain more information about the relation between A and B.

Definition 2. We say that w belongs to the class of vectors $\mathscr{P} \subset R^{2k}_+ = \{x \in R^{2k} : x \geq 0\}$ if the following two conditions hold:

(1.e) $v_k \geq u_k \geq u_{k-1} \geq v_{k-1} \geq v_{k-2} \geq \cdots \geq u_2 \geq u_1 \geq v_1$, if k is even (i.e., if $n \equiv 1 \pmod 4$).

(1.o) $v_k \geq u_k \geq u_{k-1} \geq v_{k-1} \geq v_{k-2} \geq \cdots \geq v_2 \geq v_1 \geq u_1$, if k is odd (i.e., if $n \equiv 3 \pmod 4$).

(2) $v_k - u_k \geq u_{k-1} - v_{k-1} \geq v_{k-2} - u_{k-2} \geq \cdots \geq (-1)^k(v_1 - v_1) \geq 0$.

Note that $\mathscr{P}$ is a closed subcone of the positive cone.

As we will see in Lemma 1, the eigenvector r is in the interior of $\mathscr{P}$, so there exists $\varepsilon > 0$ sufficiently small so that $rA \geq rB^\varepsilon$; this follows from applying the arguments of the proof of Proposition 10 in [GT] to those columns $B^{(j)}$ of B which differ from the corresponding column $A^{(j)}$ of A (exchanging strict for weak inequalities where necessary). Now let λ_ε be the spectral radius of B^ε, and let s_ε be a right Perron-Frobenius eigenvector: $B^\varepsilon s_\varepsilon = \lambda_\varepsilon s_\varepsilon$. Then

$$\lambda(r, s_\varepsilon) = (\lambda r, s_\varepsilon) = (rA, s_\varepsilon) \geq (rB^\varepsilon, s_\varepsilon) = (r, B^\varepsilon s_\varepsilon) = \lambda_\varepsilon(r, s_\varepsilon).$$

This implies that $\lambda \geq \lambda_\varepsilon$ since r is nonnegative and nonzero and s_ε is positive by Perron-Frobenius and the irreducibility of B^ε.

But since B^ε is irreducible and $B^\varepsilon \geq B$, we would have $\lambda_\varepsilon > \lambda$ unless $B^\varepsilon = B$. So $B^\varepsilon = B = A$ and we are finished. □

Lemma 1. *If r is a (left) Perron-Frobenius eigenvector for $A = M(\theta_n)$, then it lies in the interior of $\mathscr{P}$.*

That r lies in $\mathscr{P}$ follows easily from the invariance of $\mathscr{P}$ under the action of A^2, proved in [GT]. For the proof of Theorem 2, it is crucial that r is not on the boundary of the cone $\mathscr{P}$.

The proof of Lemma 1 relies on Lemma 2.

Lemma 2. *Let n be odd, and let $A = M(\theta_n)$. If $n \equiv 3 \pmod 4$, then*

$$(A^2)_{ij} = \begin{cases} 2\min(i, j, n-i, n-j) - 1 & \text{if } i+j=n \text{ or if } i \text{ odd} \\ & \text{and } |n-i-j| = 1; \\ 2\min(i, j, n-i, n-j) & \text{otherwise.} \end{cases}$$

If $n \equiv 1 \pmod 4$, then

$$(A^2)_{ij} = \begin{cases} 2\min(i, j, n-i, n-j) - 1 & \text{if } i+j=n \text{ or if } i \text{ even} \\ & \text{and } |n-i-j| = 1; \\ 2\min(i, j, n-i, n-j) & \text{otherwise.} \end{cases}$$

Lemma 2 can be obtained by a routine calculation, which we omit, from the following fact, found in [GT]:

Proposition 1. *Let n be odd, $k = (n-1)/2$, and $A = M(\theta_n)$. If k is odd, then*

$$A_{ij} = 1 \text{ iff } \begin{cases} k-j+2 \le i \le k+j+1 & \text{for odd } j \le k-2; \\ k-j+1 \le i \le k+j & \text{for even } j \le k-1; \\ 2 \le i \le 2k & \text{for } j = k; \\ j-k-1 \le i \le 3k-j & \text{for odd } j \ge k+2; \\ j-k \le i \le 3k-j+1 & \text{for even } j \ge k+1. \end{cases}$$

If k is even, then

$$A_{ij} = 1 \text{ iff } \begin{cases} k-j+1 \le i \le k+j & \text{for odd } j \le k-1; \\ k-j \le i \le k+j-1 & \text{for even } j \le k-2; \\ 1 \le i \le 2k-1 & \text{for } j = k; \\ j-k \le i \le 3k-j+1 & \text{for odd } j \ge k+1; \\ j-k+1 \le i \le 3k-j+2 & \text{for even } j \ge k+2. \end{cases}$$

In both cases A is symmetric.

Note also that A is irreducible since its $(k+1)$th row and column contain only ones.

For example, if $n = 11$, so that $k = 5$ and $\theta_n = (1\ 6\ 11\ 5\ 2\ 8\ 9\ 3\ 4\ 10\ 7)$, we have

$$A = \begin{pmatrix} 0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&1&0 \\ 0&1&1&1&1&1&1&1&1&0 \\ 0&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&0&0 \\ 0&0&1&1&1&1&1&1&0&0 \\ 0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0 \end{pmatrix}$$

and

$$A^2 = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 2 \\ 2 & 4 & 6 & 6 & 6 & 6 & 5 & 5 & 3 & 2 \\ 2 & 4 & 6 & 8 & 8 & 8 & 7 & 6 & 4 & 2 \\ 2 & 4 & 6 & 8 & 9 & 9 & 7 & 6 & 4 & 2 \\ 2 & 4 & 6 & 8 & 9 & 10 & 8 & 6 & 4 & 2 \\ 2 & 4 & 5 & 7 & 7 & 8 & 8 & 6 & 4 & 2 \\ 2 & 4 & 5 & 6 & 6 & 6 & 6 & 6 & 4 & 2 \\ 1 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Proof of Lemma 1. We describe here the case where $n \equiv 3 \pmod 4$, i.e., $k = (n-1)/2$ is odd. The case when k is even is handled similarly.

Let $r = (u_1, \ldots, u_{k-1}, u_k, v_k, v_{k-1}, \ldots, v_1)$ be a (left) Perron-Frobenius eigenvector for $A = M(\theta_n)$, so in particular we have that r is (strictly) positive and $rA^2 = \lambda^2 r$ with λ the spectral radius of A. We use this, together with Lemma 2 and the fact that $r \in \mathscr{P}$.

Note first that $v_1 - u_1 = \lambda^2(v_2 + v_1 - u_1) \geq \lambda^2 v_1 > 0$, and of course $u_1 > 0$. To verify that the inequalities in the first part of Definition 2 are in fact strict for r, we first observe that for $j = 1, 2, \ldots, (k-1)/2$, we have $v_{2j+1} > u_{2j+1}$ since $v_{2j+1} - u_{2j+1} \geq v_1 - u_1 > 0$, and similarly $u_{2j} - v_{2j} \geq v_1 - u_1 > 0$. To see that for such j, $v_{2j} > v_{2j-1}$ and $u_{2j+1} > u_{2j}$, it suffices to notice that the pth column of A^2 dominates the $(p-1)$th (respectively $(p+1)$th) column for $p \leq k$ (respectively $p > k$) and has some entries strictly larger.

It remains to check that the inequalities in the second part of the definition of $\mathscr{P}$ are strict for r. We consider half of these inequalities at a time. For $j = 1, 2, \ldots, (k-3)/2$, we have

$$\begin{aligned}(v_{2j+1} - u_{2j+1}) - (u_{2j} - v_{2j}) &= v_{2j} + v_{2j+1} - (u_{2j} + u_{2j+1}) \\ &= \lambda^2(2v_{2j} + v_{2j+1} + v_{2j+2} - u_{2j-1} - u_{2j} - 2u_{2j+1}) \\ &\geq \lambda^2(2v_{2j} - u_{2j-1} - u_{2j}) \\ &\geq \lambda^4(3v_{2j} - 2u_{2j-1}) \\ &> 0.\end{aligned}$$

Also,

$$\begin{aligned}v_{k-1} + v_k - (u_{k-1} + u_k) &= \lambda^2(2v_{k-1} + v_k - u_{k-2} - u_{k-1} - u_k) \\ &\geq \lambda^2(2v_{k-1} - u_{k-2} - u_{k-1}) \\ &\geq \lambda^4(3v_{k-1} - 2u_{k-2}) \\ &> 0.\end{aligned}$$

Similarly, for $j = 2, 3, \ldots, (k-1)/2$, we have

$$\begin{aligned}v_{2j-1} + u_{2j} - (v_{2j-1} + v_{2j}) &= \lambda^2(2u_{2j-1} + u_{2j} + u_{2j+1} - v_{2j-2} - v_{2j-1} - 2v_{2j}) \\ &\geq \lambda^2(2u_{2j-1} - v_{2j-2} - v_{2j-1}) \\ &\geq \lambda^4(3u_{2j-1} - 2v_{2j-2}) \\ &> 0\end{aligned}$$

and

$$\begin{aligned} u_1 + u_2 - (v_1 + v_2) &= \lambda^2(2u_1 + u_2 + u_3 - v_1 - 2v_2) \\ &\geq \lambda^4(3u_1) \\ &> 0. \quad \square \end{aligned}$$

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