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PITT'S INEQUALITY AND THE UNCERTAINTY PRINCIPLE

WILLIAM BECKNER

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ABSTRACT. The "uncertainty principle" is formulated using logarithmic estimates obtained from a sharp form of Pitt's inequality. The qualitative nature of this result underlies the relations connecting entropy, the Hardy-Littlewood-Sobolev inequality, and the logarithmic Sobolev inequality.

1. LOGARITHMIC UNCERTAINTY

Weighted inequalities for the Fourier transform provide a natural method to measure uncertainty. For functions on $\mathbb{R}^n$ the issue is the balance between the relative sizes of a function and its Fourier transform at infinity. A simple argument based on a sharp form of Pitt's inequality is used to obtain a logarithmic estimate of uncertainty.

Theorem 1. *For* $f \in \mathcal{S}(\mathbb{R}^n)$

$$(1) \qquad \int_{\mathbb{R}^n} \ln|x|\,|f(x)|^2\,dx + \int_{\mathbb{R}^n} \ln|\xi|\,|\hat{f}(\xi)|^2\,d\xi \geq D \int_{\mathbb{R}^n} |f(x)|^2\,dx,$$

$$D = \psi(n/4) - \ln\pi, \qquad \psi(t) = \frac{d}{dt}[\ln\Gamma(t)].$$

Here $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class and the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i\xi x} f(x)\,dx .$$

Since the individual terms on the left-hand side of the above expression may be indeterminate on $L^2(\mathbb{R})$, this inequality is realized as an a priori limit. The result follows from Pitt's inequality.

Theorem 2. *For* $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $0 \leq \alpha < n$

$$(2) \qquad \int_{\mathbb{R}^n} |\xi|^{-\alpha}|\hat{f}(\xi)|^2\,d\xi \leq C_\alpha \int_{\mathbb{R}^n} |x|^\alpha |f(x)|^2\,dx,$$

$$C_\alpha = \pi^\alpha \left[\Gamma\left(\frac{n-\alpha}{4}\right)\Big/\Gamma\left(\frac{n+\alpha}{4}\right)\right]^2.$$

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Proof of Theorem 1. Since inequality (2) is an equality for $\alpha = 0$, this expression can be differentiated as a function of the parameter α at that value to obtain inequality (1).

Two immediate observations concerning (1): it is dilation invariant and the left-hand side of the equation is diminished if f is replaced by its equimeasurable radial decreasing rearrangement (and similarly for $\hat{f}$). Since the logarithm is a concave function, one has for $\|f\|_2 = 1$

$$\ln\left[\int_{\mathbb{R}^n} |x|^2|f(x)|^2\,dx \int_{\mathbb{R}^n} |\xi|^2|\hat{f}(\xi)|^2\,d\xi\right]^{1/2} \geq D = \psi(n/4) - \ln\pi\,.$$

This gives a crude estimate for the Heisenberg-Weyl uncertainty principle:

$$(3)\qquad \left[\int_{\mathbb{R}^n} |x-\hat{x}|^2|f(x)|^2\,dx \int_{\mathbb{R}^n} |\xi-\hat{\xi}|^2|\hat{f}(\xi)|^2\,d\xi\right]^{1/2} \geq \frac{n}{4\pi}\int_{\mathbb{R}^n} |f(x)|^2\,dx$$

where $\hat{x}$, $\hat{\xi}$ denote mean values for their respective distributions. A consequence is that $\ln(n/4) > \psi(n/4)$ which can be checked using Jensen's inequality to show that $(\ln\beta)\Gamma(\beta) > \Gamma'(\beta)$. But in fact logarithmic uncertainty (1) implies the Heisenberg-Weyl inequality. More precisely, the calculation above gives for product functions $\prod f(x_k)$ with $\|f\|_2 = 1$

$$\ln\left[16\pi^2\int_{\mathbb{R}} |x-\hat{x}|^2|f(x)|^2\,dx \int_{\mathbb{R}} |\xi-\hat{\xi}|^2|\hat{f}(\xi)|^2\,d\xi\right]^{1/2} \geq \psi(n/4) - \ln(n/4)\,.$$

Using Gauss' expression for ψ from Whitaker and Watson ([17], pp. 247–248)

$$\psi(z) - \ln z = \int_0^\infty e^{-tz}\left[\frac{1}{t} - \frac{1}{1-e^{-t}}\right]dt \leq 0\,,$$

and since the bracket part of the integrand lies between 0 and -1, it follows that

$$-\frac{1}{z} \leq \psi(z) - \ln z \leq 0\,.$$

The asymptotic limit $n \to \infty$ then gives the value zero on the right-hand side above and reproduces the classical uncertainty principle in one dimension. By utilizing the product structure, one obtains the n-dimensional form (3). Conceptually logarithmic uncertainty can be represented in canonical form as

$$\langle \ln|Q-\hat{Q}|\rangle + \langle \ln|P-\hat{P}|\rangle \geq C\,.$$

Proof of Theorem 2. The argument to calculate C_α is simple and based on an observation about Young's inequality for convolution on some unimodular non-compact Lie groups (including Euclidean and nilpotent cases)

$$\|\psi * f\|_{L^p(G)} \leq \|\psi\|_{L^1(G)}\|f\|_{L^p(G)}.$$

For non-negative ψ with $1 < p < \infty$ this inequality is sharp with no extremal functions. Inequality (2) is dilation invariant and it is improved if f is replaced by its equimeasurable radial decreasing rearrangement. The inequality is then equivalent to a convolution inequality on the multiplicative group $\mathbb{R}_+$. This remark is implemented in two steps: first, by considering the equivalent Stein-Weiss fractional integral inequality on $\mathbb{R}^n$

$$
\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)\frac{1}{|x|^{\alpha/2}}\,\frac{1}{|x-y|^{n-\alpha}}\,\frac{1}{|y|^{\alpha/2}}f(y)\,dx\,dy\right| \le C_\alpha\left[\pi^{\frac{n}{2}-\alpha}\Gamma\left(\frac{\alpha}{2}\right)\Big/\Gamma\left(\frac{n-\alpha}{2}\right)\right]\int_{\mathbb{R}^n}|f(x)|^2\,dx, \tag{4}
$$

and since f can be taken to be a radial function, the equivalent $\mathbb{R}_+$ convolution inequality is obtained by setting $t=|x|$, $h(t)=|x|^{n/2}f(x)$:

$$
\left|\int_{\mathbb{R}_+\times\mathbb{R}_+} h(t)\psi(s/t)h(s)\frac{ds}{s}\frac{dt}{t}\right| \le C_\alpha\left[\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Big/2\pi^\alpha\Gamma\left(\frac{n-\alpha}{2}\right)\right]\int_{\mathbb{R}_+}|h(t)|^2\frac{dt}{t} \tag{5}
$$

with

$$
\psi(t)=\int_{S^{n-1}}\left[t+\frac{1}{t}-2\xi_1\right]^{-(n-\alpha)/2} d\xi
$$

where $d\xi$ denotes normalized surface measure and ξ_1 is the first component of ξ. It is useful to observe that f can be taken to be radial decreasing in (4) and h can be taken to be symmetric decreasing on $\mathbb{R}_+$ so that in addition f has an inversion symmetry.

Now the comment about Young's inequality above implies that the best constant in (5) is the L^1 norm of ψ so that

$$
\|\psi\|_{L^1(\mathbb{R}_+)}=C_\alpha\left[\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Big/2\pi^\alpha\Gamma\left(\frac{n-\alpha}{2}\right)\right]
$$

where C_α is now specified as the best constant in (2). To compute the integral of ψ, observe that

$$
\begin{aligned}
\|\psi\|_{L^1(\mathbb{R}_+)} &= \int_0^\infty\left[\int_{S^{n-1}}\left[t+\frac{1}{t}-2\xi_1\right]^{-(n-\alpha)/2} d\xi\right]\frac{dt}{t}\\
&= \left[\frac{2\pi^{n/2}}{\Gamma(n/2)}\right]^{-1}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-\alpha}}\,\frac{1}{|y|^{(n+\alpha)/2}}\,dy
\end{aligned}
$$

for $|x|=1$. Then

$$
\|\psi\|_{L^1(\mathbb{R}_+)}=\frac{\Gamma(\frac{n}{2})\Gamma(\frac{\alpha}{2})}{2\Gamma(\frac{n-\alpha}{2})}\left[\frac{\Gamma(\frac{n-\alpha}{4})}{\Gamma(\frac{n+\alpha}{4})}\right]^2
$$

so that

$$
C_\alpha=\pi^\alpha\left[\frac{\Gamma(\frac{n-\alpha}{4})}{\Gamma(\frac{n+\alpha}{4})}\right]^2 .
$$

By retracing the steps of this argument, the remark about Young's inequality and convolution implies that while C_α is sharp, no extremals exist for inequality (2).

2. Logarithmic Sobolev inequality

A stronger qualitative implication can be drawn from the logarithmic estimate (1) which connects the Hardy-Littlewood-Sobolev inequality and the logarithmic Sobolev inequality. As emphasized in [3], the logarithmic Sobolev inequality can be interpreted as sharpening the uncertainty principle. For $|f|$ radial decreasing with $\|f\|_2 = 1$ and C denoting a generic constant

$$|f(x)| \le \frac{C}{|x|^{n/2}} \quad \text{or} \quad \frac{n}{2} \ln|x| \le -\ln|f(x)| + C\,.$$

Then (1) implies

$$\frac{n}{2}\int_{\mathbb{R}^n} \ln|\xi|\,|\hat{f}(\xi)|^2\,d\xi \ge \int_{\mathbb{R}^n} \ln|f(x)|\,|f(x)|^2\,dx + C\,.$$

Since the left-hand side is a measure of smoothness, this statement is in fact a "logarithmic Sobolev inequality". Using the Hardy-Littlewood-Sobolev inequality, a sharp form of this estimate can be derived.

Theorem 3. *For $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_2 = 1$*

$$\frac{n}{2}\int_{\mathbb{R}^n} \ln|\xi|\,|\hat{f}(\xi)|^2\,d\xi \ge \int_{\mathbb{R}^n} \ln|f(x)|\,|f(x)|^2\,dx + B_n\,, \tag{6}$$

$$B_n = \frac{n}{2}\psi\left(\frac{n}{2}\right) - \frac{n}{4}\ln\pi - \frac{1}{2}\ln\left[\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right].$$

Up to conformal automorphism, extremal functions are of the form $A(1+|x|^2)^{-n/2}$.

Proof. From the sharp Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$ [8]

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)\,dx\,dy\right| \le A_p \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}\,, \tag{7}$$

$$A_p = \pi^{n/p'} \frac{\Gamma(\frac{n}{p} - \frac{n}{2})}{\Gamma(\frac{n}{p})}\left[\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right]^{\frac{2}{p}-1},$$

for $\lambda = 2n/p'$, $1 \le p < 2$ and $1/p + 1/p' = 1$, one obtains another sharp form of Pitt's inequality:

$$\int_{\mathbb{R}^n} |\xi|^{n(1-\frac{2}{p})} |\hat{f}(\xi)|^2\,d\xi \le K_p \left[\|f\|_{L^p(\mathbb{R}^n)}\right]^2, \tag{8}$$

$$K_p = \pi^{(\frac{n}{p}-\frac{n}{2})} \frac{\Gamma(\frac{n}{p'})}{\Gamma(\frac{n}{p})}\left[\frac{\Gamma(n)}{\Gamma(\frac{n}{p})}\right]^{\frac{2}{p}-1}.$$

While (7) is not defined at $p = 2$, (8) is an equality at this point and so can be differentiated to produce inequality (6). The conformal invariance of this inequality is inherited from the Hardy-Littlewood-Sobolev inequality and this suffices to determine the extremals.

The next step is to relate the estimate (6) to the standard logarithmic Sobolev inequality for Gaussian measure:

$$\int_{\mathbb{R}^n} |g|^2 \ln|g|\,d\mu \le \int_{\mathbb{R}^n} |\nabla g|^2\,d\mu \tag{9}$$

where

$$d\mu = (2\pi)^{-n/2} e^{-x^2/2}\,dx\,, \qquad \|g\|_{L^2(d\mu)} = 1\,.$$

Setting $f(x) = (2\pi)^{-n/4} g(x) e^{-x^2/4}$, one finds from (9) for $\|f\|_2 = 1$

$$\frac{n}{2} + \frac{n}{4} \ln(2\pi) + \int_{\mathbb{R}^n} |f|^2 \ln |f| \, dx \le \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \, .$$

Now this inequality is not dilation-invariant so setting up a variational problem in terms of a dilation parameter, one finds the estimate

$$\frac{n}{4} \ln \left(\frac{n\pi e}{2} \right) + \int_{\mathbb{R}^n} |f|^2 \ln |f| \, dx \le \frac{n}{4} \ln \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \tag{10}$$

or equivalently

$$\frac{n}{4} \ln \left(\frac{ne}{8\pi} \right) + \int_{\mathbb{R}^n} |f|^2 \ln |f| \, dx \le \frac{n}{4} \ln \int_{\mathbb{R}^n} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi \, . \tag{11}$$

Using Jensen's inequality the qualitative nature of this last estimate can be obtained from the logarithmic uncertainty principle (1). In addition, there is a nice interpretation of inequality (10) in terms of entropy and Fisher information. For a probability density φ on $\mathbb{R}^n$ denote the entropy and information functionals respectively by

$$E(\varphi) = -\int_{\mathbb{R}^n} \varphi \ln \varphi \, dx \, , \qquad I(\varphi) = \int_{\mathbb{R}^n} |\nabla \varphi|^2 \frac{1}{\varphi} \, dx \, .$$

Then (10) can be rewritten as

$$E(\varphi) + \frac{n}{2} \ln \left[\frac{1}{2\pi n e} I(\varphi) \right] \ge 0 \, . \tag{12}$$

Using the classical entropy inequality for a probability density

$$E(\varphi) \le \frac{n}{2} \ln \left[\frac{2\pi e}{n} Var(\varphi) \right] \, ,$$

one finds the Cramér-Rao form of the Heisenberg uncertainty inequality (3)

$$I(\varphi) \; Var(\varphi) \; \ge \; n^2 ,$$

which demonstrates that the logarithmic Sobolev inequality determines the classical uncertainty principle.

It is very natural to link fractional integration to logarithmic Sobolev inequalities. This principle was applied in [2] to obtain logarithmic estimates on the n-dimensional sphere from the Hardy-Littlewood-Sobolev inequality and prove corresponding hypercontractive estimates for the heat and Poisson semigroups on the sphere. Here a simple asymptotic estimate is used to derive the logarithmic Sobolev inequality (9) as a consequence of Theorem 3. Replacing estimate (11) by the analogous estimate implied by (6) and reversing the steps used to get (11), one finds

$$W_n + \int_{\mathbb{R}^n} |g|^2 \ln |g| \, d\mu \le \int_{\mathbb{R}^n} |\nabla g|^2 \, d\mu , \tag{13}$$

$$W_n = B_n - \frac{n}{4} \ln \left(\frac{ne}{8\pi} \right) = \frac{n}{2} \psi \left(\frac{n}{2} \right) - \frac{1}{2} \ln \left[\frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right] - \frac{n}{4} \ln \left(\frac{ne}{8} \right) \le 0$$

where the $L^2(d\mu)$ norm of g is one. Fix $m \ll n$ and consider functions g which are constant in the variables x_k, $(m+1) \le k \le n$. Then

$$W_n + \int_{\mathbb{R}^m} |g|^2 \ln |g| \, d\mu \le \int_{\mathbb{R}^m} |\nabla g|^2 \, d\mu$$

holds for all n and it suffices to find the limit of W_n as $n \to \infty$. Using the expression for ψ taken from Whitaker and Watson ([17], p. 251)

$$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi t}-1)}\, dt,$$

$$\lim_{n\to\infty} W_n = -\frac{1}{2} + \frac{1}{4}\ln 2;$$

then

$$-\frac{1}{2} + \frac{1}{4}\ln 2 + \int_{\mathbb{R}^m} |g|^2 \ln |g| \, d\mu \le \int_{\mathbb{R}^m} |\nabla g|^2 \, d\mu . \tag{14}$$

Taking $g(x_1, \dots, x_m) = \prod h(x_k)$, one has

$$\frac{1}{m}\left[-\frac{1}{2}\ln\left(\frac{e}{\sqrt{2}}\right)\right] + \int_{\mathbb{R}} |h|^2 \ln |h| \, d\mu \le \int_{\mathbb{R}} |\nabla h|^2 \, d\mu$$

and the resulting limit $m \to \infty$ produces the one-dimensional logarithmic Sobolev inequality. Gaussian symmetry or convexity then extends the inequality to all dimensions.

3. Entropy and the Fourier transform

The calculations for the logarithmic Sobolev inequality illustrate that entropy and the class $L\log L$ are central to developing geometric and probabilistic information from sharp function space estimates. The logarithmic estimate (1) is maximized on the class $\zeta(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : f, \hat{f}$ non-negative radial decreasing functions$\}$. For this class one observes that (1) implies for $\|f\|_2 = 1$

$$-\int_{\mathbb{R}^n} \ln |f(x)| \, |f(x)|^2 \, dx - \int_{\mathbb{R}^n} \ln |\hat{f}(\xi)| \, |\hat{f}(\xi)|^2 \, d\xi \ge C.$$

Such an estimate holds in general on $\mathcal{S}(\mathbb{R}^n)$ with the best value of C being attained for Gaussian functions. This result was given in [1] by differentiating the sharp Hausdorff-Young inequality at $p = 2$.

Theorem 4. *For* $f \in \mathcal{S}(\mathbb{R}^n)$ *with* $\|f\|_2 = 1$

$$-\int_{\mathbb{R}^n} \ln |f(x)| \, |f(x)|^2 \, dx - \int_{\mathbb{R}^n} \ln |\hat{f}(\xi)| \, |\hat{f}(\xi)|^2 \, d\xi \ge \frac{n}{2}(1 - \ln 2). \tag{15}$$

For problems where Gaussian structure has an intrinsic role, it is always useful to make explicit the interplay between different realizations determined by Gaussian and Lebesgue measures. Here the entropy estimate for the Fourier transform produces a surprising strengthening of the logarithmic Sobolev inequality using only a simple change of variables.

Theorem 5. *For* $g \in L^2(d\mu)$, $d\mu = (2\pi)^{-n/2}\exp(-x^2/2)\,dx$ *and* $\|g\|_2 = 1$

$$\int_{\mathbb{R}^n} |g|^2 \ln|g|\,d\mu + \int_{\mathbb{R}^n} |\tilde{g}|^2 \ln|\tilde{g}|\,d\mu \le \int_{\mathbb{R}^n} |\nabla g|^2\,d\mu \tag{16}$$

where $\tilde{g}$ *is defined by a linear action on* g *in terms of the standard representation of the Hermite polynomials* $\{H_n\}$ *for the Gaussian measure* $d\mu$: $\widetilde{H}_n(x_k) = i^n H_n(x_k)$.

Proof. Set $f(x) = 2^{n/4} g(2\sqrt{\pi}\,x)\exp(-\pi|x|^2)$; then

$$\hat{f}(\xi) = 2^{n/4}\tilde{g}(2\sqrt{\pi}\,\xi)\exp(-\pi|\xi|^2).$$

Substitute these forms in equation (15). Using the fact that for the one-dimensional harmonic oscillator with N denoting the number operator corresponding to the measure $d\mu$,

$$H = \left(-\frac{d^2}{dx^2} + \frac{1}{4}x^2\right) = \frac{1}{2} + N,$$

then this spectral relation is realized as

$$\frac{1}{4}\int_{\mathbb{R}^n} |x|^2 |g(x)|^2\,d\mu + \frac{1}{4}\int_{\mathbb{R}^n} |\xi|^2 |\tilde{g}(\xi)|^2\,d\mu = \frac{n}{2} + \int_{\mathbb{R}^n} |\nabla g|^2\,d\mu \tag{17}$$

and inequality (16) is obtained.

Theorem 4 may be regarded as stating the Hausdorff-Young inequality for entropy

$$E(|f|^2) + E(|\hat{f}|^2) \ge n\ln(\frac{e}{2})$$

where $\|f\|_2 = 1$. Using the classical inequality that relates entropy to variance for a probability density, one finds as in [1] that the sharp Hausdorff-Young inequality implies the Heisenberg uncertainty principle (3)

$$\ln(\frac{n}{4\pi})^2 \le \ln\left[Var(|f|^2)\right] + \ln\left[Var(|\hat{f}|^2)\right].$$

The context of this interplay between the Hausdorff-Young inequality and the standard representation of quantum mechanics has contributed to the interest in "sharp constants".

Remarks. 1. The logarithmic uncertainty inequality (1) is not sharp for Gaussian functions, but interestingly it is asymptotically sharp on this class. For Gaussian functions

$$\int_{\mathbb{R}^n} \ln|x|\,|f(x)|^2\,dx + \int_{\mathbb{R}^n} \ln|\xi|\,|\hat{f}(\xi)|^2\,d\xi = [\psi(n/4) - \ln\pi + \beta(n/2)]\int_{\mathbb{R}^n} |f|^2\,dx$$

where

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt$$

and $\beta(x) \to 0$ as $x \to \infty$.

2. The nature of the sharp n-dimensional constant D_n for logarithmic uncertainty is determined by the Euclidean product structure. Using the arithmetic-geometric mean estimate

$$\ln|x| \ge \ln\sqrt{n} + \frac{1}{n}\sum \ln|x_k|,$$

it follows that $D_n \geq \ln n + D_1$ which gives the correct asymptotic growth. From the previous integral representations for ψ, one finds that $\psi'(z) \geq 1/z$ which can be used to check this lower bound for D_n.

3. Inequality (10) first appeared in Weissler's paper on the heat-diffusion semigroup [15]. The simple argument using dilation invariance occurs in the author's research note "Entropy and Sobolev Inequalities" written in 1983 as does the interpretation in terms of Fisher information and entropy (12) and the strengthened logarithmic Sobolev inequality (16) obtained from the entropy inequality for the Fourier transform. The proof there for (16) used a differentiation argument applied to sharp estimates for the Hermite semigroup. A recent paper by E. Carlen [J. Funct. Anal. **101** (1991), 194–211] develops similar results. Weissler conjectured that inequality (10) might be obtained from Nirenberg's form of the Sobolev inequalities with sharp constants. In one dimension this argument can be carried out using variational inequalities in Lieb [8] and Nagy [Acta Sci. Math. (Szeged) **10** (1941), 64–74].

4. In view of Theorem 4, it is natural to ask what information can be obtained by taking the limit of the sharp Hausdorff-Young inequality near L^1. In this case one simply recovers the classical inequality relating entropy to variance for a probability density.

5. The background for Pitt's inequality, especially the form (8), in terms of the Hardy-Littlewood and Paley theorems for Fourier coefficients is described in Zygmund [18]. The Stein-Weiss integral (4) arises in the analysis of complementary series representations for the Lorentz group. A broader framework for representing the uncertainty principle is given in Folland's recent treatise [4] (also see chapter 12 in [12]). Inequalities in the context of the Heisenberg group will be discussed in a forthcoming paper. In extending weighted inequalities to more geometric manifolds, it becomes evident that Pitt's inequality and the uncertainty principle are quantitative statements about the dilation structure of the manifold.

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Department of Mathematics, University of Texas at Austin, Austin, Texas 78712
E-mail address: beckner@math.utexas.edu