PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

CONDITION $\mathscr{B}$ AND BAIRE 1 GENERALIZED DERIVATIVES

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(Communicated by Andrew Bruckner)

ABSTRACT. Ordered pairs (F, f) of real-valued functions on $[0, 1]$ which satisfy the condition that every perfect set M contains a dense G_δ set K such that $F|M$ is differentiable to f on K are shown to play a key role in several types of generalized differentiation. In particular, this condition is utilized to prove the equivalence of selective differentiation and various forms of path differentiation under the assumption that the derivatives involved are of Baire class 1, thereby providing an affirmative answer, for Baire 1 selective derivatives, to a question raised in [Trans. Amer. Math. Soc **283** (1984), 97–125].

Throughout this paper F and f will denote real-valued functions defined on $[0, 1]$. We shall say that the ordered pair of functions (F, f) satisfies *Condition* $\mathscr{B}$ if every perfect set M contains a dense G_δ set K such that $F|M$ is differentiable to f on K. This condition plays a key role in situations where F is selectively differentiable to f or path differentiable to f.

Selective differentiation was introduced by the third author in [3]. Unlike the situation with many generalized derivatives, it was observed there that selective derivatives need not belong to Baire class one. M. Laczkovich [2] showed that they must belong to Baire class two, however. One consequence of the present paper is that if f is a Baire 1 function, then a function F is selectively differentiable to f if and only if f is a bilateral derivate function of F and (F, f) satisfies Condition $\mathscr{B}$.

Path differentiation was introduced by Bruckner, Thomson, and the third author in [1], where it was found that many properties of functions and their path derivatives are based on various intersection conditions and it was asked if there is an equivalence between selective derivatives and one type of path derivatives. Utilizing Condition $\mathscr{B}$, we shall answer this question affirmatively in the situation where the derivative in question is in Baire 1. An unexpected equivalence among various path derivatives is also established.

To clarify the technical terms utilized above, we need to review some terminology from [3] and [1]. We will use the notation $[a, b]$, or (a, b), to denote the closed, or open, interval having endpoints a and b regardless of

Received by the editors September 15, 1993.
1991 *Mathematics Subject Classification*. Primary 26A24.
Key words and phrases. Baire class 1, selections, path systems, differentiation, Condition $\mathscr{B}$.

0002-9939/95 \$1.00 + \$.25 per page

whether $a > b$ or $b > a$. A *selection function* is obtained by assigning to each closed interval $[a, b]$ in $[0, 1]$ a point from (a, b) and labeling it $p_{[a,b]}$. The collection of p's thus obtained is called a *selection* S. We say that F is *selectively differentiable to* f on $[0, 1]$ if there is a selection S such that for each $x \in [0, 1]$

$$\lim_{y \to x} \frac{F(p_{[x,y]}) - F(x)}{p_{[x,y]} - x} = f(x).$$

Next, let $x \in [0, 1]$. A *path leading to* x is a set $E_x \subseteq [0, 1]$ containing x and having x as an accumulation point. A *path system* is a collection $E = \{E_x : x \in [0, 1]\}$ such that each E_x is a path leading to x. If for each $x \in (0, 1)$, E_x has x as a bilateral limit point, then E is called a *bilateral path system.* We say that F is path differentiable to f if there is a path system E such that for each $x \in [0, 1]$

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y) - F(x)}{y - x} = f(x).$$

In [1] the following four types of intersection properties were investigated. A system of paths E is said to satisfy the conditions listed below if there is associated with E a positive function δ on $[0, 1]$ such that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$, then E_x and E_y intersect in the stated fashion:

- *intersection condition (I.C.)*: $E_x \cap E_y \cap [x, y] \neq \emptyset$;
- *internal intersection condition (I.I.C.)*: $E_x \cap E_y \cap (x, y) \neq \emptyset$;
- *external intersection condition, parameter* $m > 0$ *(E.I.C.[m])*:

$$E_x \cap E_y \cap (y, (m+1)y - mx) \neq \emptyset \text{ and } E_x \cap E_y \cap ((m+1)x - my, x) \neq \emptyset;$$

- *one-sided external intersection condition, parameter* $m > 0$ *(one-sided E.I.C.[m])*:

$$E_x \cap E_y \cap (y, (m+1)y - mx) \neq \emptyset \text{ or } E_x \cap E_y \cap ((m+1)x - my, x) \neq \emptyset.$$

A statement such as "F is I.C.(bilaterally)-path differentiable to f" will indicate that F is path differentiable to f with respect to a (bilateral) path system which satisfies I.C. It is clear that if F is I.I.C.-path differentiable to f, then it is I.C.-path differentiable to f. Likewise, if F is E.I.C.[m]-path differentiable to f, then it is one-sided E.I.C.[m]-path differentiable to f. Theorem 3.4 in [1] shows that if f is a bilateral derivate function of F and F is I.I.C.-path differentiable to f, then F is selectively differentiable to f. We shall show that the two notions are actually equivalent when f is Baire 1.

Lemma 1. *Suppose that the function* $F : [0, 1] \to \mathbb{R}$ *is selectively differentiable to* f *on* $[0, 1]$ *or path differentiable to* f *on* $[0, 1]$ *for some path system satisfying any one of the four intersection properties. Furthermore, assume that* $0 < \epsilon < 1$ *and* M *is a closed subset of* $[0, 1]$ *such that* F *is Lipschitz on* M *and* $|f(x) - f(y)| < \epsilon$ *for all* x *and* y *in* M. *Then there exists a constant* c, *independent of* ϵ *and* M, *and a set* $U \subseteq M$, *relatively open in* M, *such that for all distinct* x *and* y *in* U

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < c\epsilon. \tag{1}$$

Proof. To prove all five statements contained in this lemma, it clearly suffices to prove the result only in the three situations where F is selectively differentiable to f, I.C.-path differentiable to f, and one-sided E.I.C.[m]-path differentiable to f, since each of the other two hypotheses implies one of these three. First we consider the situation where F is selectively differentiable to f. Let S denote the given selection with $p_{[x,y]}$ denoting the selected point from $[x, y]$. For each $n \in \mathbb{N}$ let

$$A_n = \left\{ x \in M : \forall y \text{ satisfying } 0 < |x-y| < \frac{1}{n}, \left| \frac{F(p_{[x,y]}) - F(x)}{p_{[x,y]} - x} - f(x) \right| < \epsilon \right\}.$$

By the Baire Category Theorem there exists an $n \in \mathbb{N}$ and a relatively open set U in M such that A_n is dense in U. We may assume that the diameter of U is less than $1/n$.

We wish to establish (1) for all distinct x and y in U. First, suppose that $u, v \in U \cap A_n$. Then

$$\begin{aligned}\left|\frac{F(u)-F(v)}{u-v} - f(v)\right| &\le \left|\frac{F(u)-F(p_{[u,v]})}{u-p_{[u,v]}} - f(u)\right| \cdot \left|\frac{u-p_{[u,v]}}{u-v}\right| \\ &\quad + \left|\frac{F(p_{[u,v]})-F(v)}{p_{[u,v]}-v} - f(v)\right| \cdot \left|\frac{p_{[u,v]}-v}{u-v}\right| \\ &\quad + |f(u)-f(v)| \cdot \left|\frac{u-p_{[u,v]}}{u-v}\right| \\ &< \epsilon \cdot 1 + \epsilon \cdot 1 + \epsilon \cdot 1 \\ &= 3\epsilon. \end{aligned} \tag{2}$$

Now let $x, y \in U$. Let $L > 1$ denote a constant which is greater than both the bound on $|f|$ on M and the Lipschitz constant for F on M. Since A_n is dense in U, we may find $u, v \in A_n \cap U$ for which both $|x-v|$ and $|y-u|$ are less than $\epsilon|x-y|/L$. Then, utilizing (2), we obtain

(3)
$$\begin{aligned}\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &\le \left|\frac{F(y)-F(u)}{y-u}\right| \cdot \left|\frac{y-u}{y-x}\right| + \left|\frac{F(v)-F(x)}{v-x}\right| \cdot \left|\frac{v-x}{y-x}\right| \\ &\quad + \left|\frac{F(u)-F(v)}{u-v} - f(v)\right| \cdot \left|\frac{u-v}{y-x}\right| \\ &\quad + |f(x)-f(v)| \cdot \left|\frac{u-v}{y-x}\right| + |f(x)| \cdot \frac{|v-x|+|y-u|}{|y-x|} \\ &< L \cdot \frac{\epsilon}{L} + L \cdot \frac{\epsilon}{L} + 3\epsilon \cdot 3 + \epsilon \cdot 3 + L \cdot \frac{2\epsilon}{L} \\ &= 16\epsilon, \end{aligned}$$

completing the proof for the selective differentiation case.

Next, suppose that F is I.C.-path differentiable to f or one-sided E.I.C.[m]-path differentiable to f, and for each x let E_x denote its path. This time for each $n \in \mathbb{N}$ we let

$$A_n = \left\{ x \in M : \delta(x) > \frac{1}{n} \ \& \ \forall y \in E_x , \right.$$
$$\left. 0 < |x-y| < \frac{1}{n}, \left| \frac{F(y)-F(x)}{y-x} - f(x) \right| < \epsilon \right\}.$$

By the Baire Category Theorem there exists an $n \in \mathbb{N}$ and a relatively open set U in M such that A_n is dense in U. We may assume that the diameter of U is less than $1/n$.

Dealing first with the case where F is I.C.-path differentiable to f, we consider any two points $u < v$ in $A_n \cap U$, and let $t \in E_u \cap E_v \cap [u, v]$. If $t \in (u, v)$, then

$$\begin{aligned} \left| \frac{F(u)-F(v)}{u-v} - f(v) \right| &\leq \left| \frac{F(u)-F(t)}{u-t} - f(u) \right| \cdot \left| \frac{u-t}{u-v} \right| \\ &\quad + \left| \frac{F(t)-F(v)}{t-v} - f(v) \right| \cdot \left| \frac{t-v}{u-v} \right| \\ &\quad + |f(u)-f(v)| \cdot \left| \frac{u-t}{u-v} \right| \\ &< \epsilon \cdot 1 + \epsilon \cdot 1 + \epsilon \cdot 1 \\ &= 3\epsilon , \end{aligned} \tag{4}$$

and if $t = u$ or $t = v$, then clearly

$$\left| \frac{F(u)-F(v)}{u-v} - f(v) \right| < \epsilon < 3\epsilon. \tag{5}$$

Noting that inequalities (4) and (5) are identical to (2), we may establish (1) for arbitrary x and y in U by proceeding exactly as in the selective case via inequality (3).

Finally, dealing with the case where F is one-sided E.I.C.[m]-path differentiable to f, we let $u < v$ be any two points in $A_n \cap U$, and let $t \in E_u \cap E_v \cap ((m+1)u - mv, u)$ or $t \in E_u \cap E_v \cap (v, (m+1)v - mu))$, depending on which intersection is nonempty. Then we have

$$\begin{aligned} \left| \frac{F(u)-F(v)}{u-v} - f(v) \right| &\leq \left| \frac{F(u)-F(t)}{u-t} - f(u) \right| \cdot \left| \frac{u-t}{u-v} \right| \\ &\quad + \left| \frac{F(t)-F(v)}{t-v} - f(v) \right| \cdot \left| \frac{t-v}{u-v} \right| \\ &\quad + |f(u)-f(v)| \cdot \left| \frac{u-t}{u-v} \right| \\ &< \epsilon \cdot (m+1) + \epsilon \cdot (m+1) + \epsilon \cdot (m+1) \\ &= (3m+3)\epsilon. \end{aligned} \tag{6}$$

Using (6) in place of (2) we may now proceed as in the selective case to show that for each distinct x and y in U

$$\left|\frac{F(y)-F(x)}{y-x}-f(x)\right|<(9m+16)\epsilon,$$

completing the proof of the lemma. □

Theorem 1. *If F has the Baire 1 function f as a selective derivative or path derivative with respect to a path system satisfying any one of the four intersection properties, then (F, f) satisfies Condition $\mathscr{B}$.*

Proof. Let F be differentiable to the Baire 1 function f in any one of the stated senses. Let M be a perfect subset of $[0, 1]$. Let $\mathscr{W}$ be a countable basis for the relative topology on M. Using either Lemma 3 from [3] for the selective case or Theorem 4.6 (see proof) from [1] for the remaining cases and the fact that f is a Baire 1 function, for each $W \in \mathscr{W}$ and for each $n \in \mathbb{N}$ we can obtain a closed set $V_n^W \subseteq W$ with nonempty interior in M such that F is Lipschitz on V_n^W and for all x and y in V_n^W we have that $|f(x)-f(y)|<\frac{1}{n}$. Now, using Lemma 1 we may find nonempty sets $U_n^W \subseteq V_n^W$ which are open relative to M such that for all distinct $x, y \in U_n^B$,

$$\left|\frac{F(x)-F(y)}{x-y}-f(x)\right|<\frac{c}{n},$$

where c is the constant from the lemma. Let $K=\bigcap_{n=1}^{\infty}\bigcup_{W\in\mathscr{W}}U_n^W$. Then, it is easily seen that $F|M$ is differentiable to f on the dense G_δ set K, completing the proof of the theorem. □

We remark that by Corollary 6.3 in [1] the Baire 1 assumption on f in the E.I.C[m] case of the previous theorem is superfluous.

Lemma 2. *Suppose that (F, f) satisfies Condition $\mathscr{B}$ and that f is a Baire 1 function. Then, it follows that for each zero-dimensional perfect set $M \subseteq [0, 1]$ and for each $\epsilon > 0$ there exists a sequence $\{C_i\}_{i=1}^{\infty}$ of disjoint closed sets such that:*

(i) $\bigcup_{i=1}^{\infty} C_i = M$.

(ii) *For each i and each distinct x and y in C_i*

$$\left|\frac{F(x)-F(y)}{x-y}-f(x)\right|<\epsilon.$$

(iii) *For each i,* $\operatorname{diam}(C_i)<\epsilon$.

Proof. Let M be a zero-dimensional perfect set in $[0, 1]$, and let $\epsilon > 0$. We first establish the following:

> CLAIM. *Let P be a perfect subset of M. Then there exists a set U which is open relative to P such that for all distinct x and y in U*
>
> $$\left|\frac{F(x)-F(y)}{x-y}-f(x)\right|<\epsilon.$$

Let P be a perfect subset of M. Since f is a Baire 1 function, f restricted to P has a point of continuity. Hence there is a portion (relatively open set)

P_1 of P such that for all $s, t \in P_1$ we have $|f(s) - f(t)| < \epsilon/4$. Furthermore, since P is zero dimensional, we may take P_1 to be closed relative to P as well as open relative to P.

For each natural number n, let

$$A_n = \left\{ s \in P_1 : \left| \frac{F(s) - F(t)}{s - t} - f(s) \right| < \frac{\epsilon}{4} \text{ for all } t \in P_1,\ 0 < |s - t| < \frac{1}{n} \right\}.$$

Recalling that $F|P_1$ is differentiable to f on a dense G_δ subset of P_1 and applying the Baire Category Theorem, we conclude that one of these A_n's, say A_m, is categorically dense in an open set relative to P_1, and hence relative to P. Let U be an open set relative to P, having diameter less than $1/m$ which is a subset of this open set.

Let x and y be distinct points in U. Choose $s \in A_m \cap U$ so close to x that $|x - s| < |x - y|/2$. Then

$$\begin{aligned} \left| \frac{F(x) - F(y)}{x - y} - f(x) \right| &\le \left| \frac{F(x) - F(s)}{x - s} - f(s) \right| \cdot \left| \frac{x - s}{x - y} \right| \\ &\quad + \left| \frac{F(s) - F(y)}{s - y} - f(s) \right| \cdot \left| \frac{s - y}{x - y} \right| + |f(s) - f(x)| \\ &< \frac{\epsilon}{4} \cdot \frac{1}{2} + \frac{\epsilon}{4} \cdot \frac{3}{2} + \frac{\epsilon}{4} < \epsilon, \end{aligned}$$

completing the proof of the claim.

To complete the proof of the lemma, we first apply the claim to $P = M$, obtaining an open set U; then we express $M \setminus U$ as the disjoint union of a perfect set P^* and a union of a countable collection of singletons; we apply the lemma to P^*; etc. This procedure can clearly be continued transfinitely. However, the process must terminate at a countable ordinal, yielding a countable decomposition of M into closed sets which satisfy conditions (i) and (ii) described in the lemma statement. Since M is zero dimensional, we may alter these closed sets so that condition (iii) is satisfied as well. □

Theorem 2. *If f is a Baire 1 derivate function of F, and (F, f) satisfies Condition $\mathscr{B}$, then*

(1) *F is I.C.-path differentiable to f. If, in addition, f is a bilateral derivate function of F, then*

(2) *F is I.I.C.-path differentiable to f, and, consequently, selectively differentiable to f;*

(3) *F is one-sided E.I.C.[m]-path differentiable to f.*

Proof. Essential to this proof is a tree technique similar to one developed by the first two authors in [4]. Let $\mathscr{I}$ denote the collection of all finite sequences ν of natural numbers, and let $\mathscr{I}^*$ denote the collection of all infinite sequences ν of natural numbers. We are going to define a collection of closed sets

$$\mathscr{G} = \{G_\nu : \nu \in \mathscr{I}\}.$$

We shall denote the length of a $\nu \in \mathscr{I}$ by $|\nu|$. We denote the kth term of a ν in either $\mathscr{I}$ or $\mathscr{I}^*$ by $\nu(k)$, and if $\nu \in \mathscr{I}$ has length at least n or if $\nu \in \mathscr{I}^*$, we let $\nu|_n$ denote the truncated sequence $\{\nu(1), \nu(2), \ldots, \nu(n)\}$. (For any ν, $\nu|_0$ will denote the empty sequence.) If $\nu \in \mathscr{I}$ and $\tau = \nu|_n$ for some n, then we say that ν is an extension of τ. (Every $\tau \in \mathcal{S}$ is considered an extension of the empty sequence.) Finally, if $|\nu| = n$ and i is a natural number, we let νi denote the sequence $\{\nu(1), \nu(2), \dots, \nu(n), i\}$. In order to define our collection $\mathscr{G}$ we first let T be an F_σ first category set such that F is differentiable to f on the complement of T. Applying Lemma 2 we find that for each natural number k there is a sequence $\{C_i^k\}_{i=1}^\infty$ of pairwise disjoint closed sets such that :

- $\bigcup_{i=1}^\infty C_i^k = T$.
- For each i and each distinct x and y in C_i^k
$$\left|\frac{F(x)-F(y)}{x-y} - f(x)\right| < \frac{1}{2^k}.$$
- $\operatorname{diam}(C_i^k) < \frac{1}{2^k}$.

Now for each $n \in \mathbb{N}$ and each ν of length n we set

$$G_\nu = \bigcap_{k=1}^n C_{\nu(k)}^k,$$

and observe that

(1) Each G_ν is a closed set of diameter less than $\frac{1}{2^{|\nu|}}$.
(2) For each natural number n, $\bigcup_{|\nu|=n} G_\nu = T$.
(3) If $\nu \neq \tau$, and neither is an extension of the other, then $G_\nu \cap G_\tau = \emptyset$.
(4) If τ is an extension of ν, then $G_\tau \subseteq G_\nu$.
(5) For each n, if $|\nu| = n$, then for each distinct x and y in G_ν
$$\left|\frac{F(x)-F(y)}{x-y} - f(x)\right| < \frac{1}{2^n}.$$

Note also that for each $x \in T$ there is an unique $\nu_x \in \mathcal{S}^*$ such that

$$\{x\} = \bigcap_{n=1}^\infty G_{\nu_x|_n}.$$

If x and y are in T and $x \neq y$, then $\nu_x \neq \nu_y$. For such x and y we let $n_{[x,y]}$ denote the smallest $n \in \mathbb{N}$ for which $\nu_x(n) \neq \nu_y(n)$. For any two points $x < y$ in $[0, 1]$ we are going to designate one as the *preferred point* as follows: if neither x nor y belong to T, then x is preferred; if exactly one of x and y belongs to T, then the one that belongs to T is preferred; and if both x and y belong to T, then x is preferred if $\nu_x(n_{[x,y]}) < \nu_y(n_{[x,y]})$, otherwise y is preferred.

Also for each $x \in [0, 1]$ we select a sequence $\{s_k(x)\}$ converging to x as follows:

- If $f(x)$ is a bilateral derivate of F at x, then let $\{s_k(x)\}$ be chosen so that
  - The sequence $\{s_k(x) - x\}$ alternates in sign.
  - The sequence $\{|s_k(x) - x|\}$ is strictly decreasing with limit 0.
  - $\lim_{k\to\infty} \frac{F(s_k(x))-F(x)}{s_k(x)-x} = f(x)$.

- If $f(x)$ is only a one-sided derivate of F at x, then let $\{s_k(x)\}$ be chosen so that
  - The sequence $\{s_k(x) - x\}$ strictly monotonically converges to 0.
  - $\lim_{k\to\infty} \frac{F(s_k(x))-F(x)}{s_k(x)-x} = f(x)$.

We now define the required path systems. Our method is based on two weakened forms of a selection function. Utilizing one type of "selection" function, we will first define paths that are I.C. or I.I.C. depending on whether f is a derivate or bilateral derivate of F. Then for a fixed $m > 0$ we will define a path system that satisfies E.I.C.[m] in a similar fashion using a somewhat different type of "selection" function.

Turning to the I.C. and I.I.C. cases first, to each pair of points $0 \le x < y \le 1$ we assign a point $q_{[x,y]} \in [x, y]$ in the following manner: For $x < y$ let u be the preferred point of x and y, and let v be the other one. If $\{s_k(u)\} \cap (x, y) \neq \emptyset$, then choose $q_{[x,y]}$ to be the first element of the sequence $\{s_k(u)\}$ in (x, y) satisfying both

(a) $|q_{[x,y]} - u| < |q_{[x,y]} - v|$, and

(b) $\left|\frac{F(q_{[x,y]}-F(u)}{q_{x,y]}-u} - f(u)\right| < |x - y|$,

and if $\{s_k(u)\} \cap (x, y) = \emptyset$, choose $q_{[x,y]} = u$. Now set

$$E_x^+ = \bigcup_{t\in(x,1]} \{q_{[x,t]}\}, \quad E_x^- = \bigcup_{t\in[0,x)} \{q_{[t,x]}\}, \quad \text{and} \quad E_x = \{x\} \cup E_x^+ \cup E_x^-.$$

It follows that E_x is a path leading to x, and since for each $0 \le x < y \le 1$ we have $q_{[x,y]} \in E_x \cap E_y \cap [x, y]$, the resulting path system $E = \{E_x : x \in [0, 1]\}$ satisfies I.C. Furthermore, if f is a bilateral derivate of F, then E satisfies I.I.C.

Suppose that x and y are in T and $n_{[x,y]} > 1$. Let u be the preferred point of x and y, and v be the other point. Then, we have that

$$\begin{aligned}
\left|\frac{F(q_{[x,y]}) - F(v)}{q_{[x,y]} - v} - f(v)\right| &\le \left|\frac{F(q_{[x,y]}) - F(u)}{q_{[x,y]} - u} - f(u)\right| \cdot \left|\frac{q_{[x,y]} - u}{q_{[x,y]} - v}\right| \\
&\quad + \left|\frac{F(u) - F(v)}{u - v} - f(v)\right| \cdot \left|\frac{u - v}{q_{[x,y]} - v}\right| \\
&\quad + |f(u) - f(v)| \cdot \left|\frac{q_{[x,y]} - u}{q_{[x,y]} - v}\right| \\
&< |x - y| \cdot 1 + \frac{1}{2^{n_{[x,y]}-1}} \cdot 2 + \frac{2}{2^{n_{[x,y]}-1}} \cdot 1 \\
&< \frac{5}{2^{n_{[x,y]}-1}}.
\end{aligned} \tag{7}$$

Next, we define an E.I.C.[m]-path system D. For each x we set $r_{[x,x]} = x$. For $0 \le x < y \le 1$ the point selected corresponding to $[x, y]$ will fall outside the interval, but close enough to it so that one-sided E.I.C.[m] will hold. Let $x < y$ be any two points in [0, 1]. We assign a point $r_{[x,y]}$ in either $(y, (m+1)y - mx) \cap (y, y + \frac{y-x}{2})$ or $((m+1)x - my, x) \cap (x - \frac{y-x}{2}, x)$ as follows: Let u be the preferred point of x and y, and v be the other point. If $u = x$, then let $r_{[x,y]}$ be the first element of the sequence $\{s_k(u)\}$ in $((m+1)x - my, x) \cap (x - \frac{y-x}{2}, x)$ satisfying

$$\left| \frac{F(r_{[x,y]}) - F(u)}{r_{[x,y]} - u} - f(u) \right| < |x-y|. \tag{8}$$

If $u = y$, then let $r_{[x,y]}$ be the first element of the sequence $\{s_k(u)\}$ in $(y, (m+1)y - mx) \cap (y, y + \frac{y-x}{2})$ satisfying (8). Then set

$$D_x^+ = \bigcup_{t \in (x,1]} \{r_{[x,t]}\}, \quad D_x^- = \bigcup_{t \in [0,x)} \{r_{[t,x]}\}, \quad \text{and} \quad D_x = \{x\} \cup D_x^+ \cup D_x^-.$$

It follows that D_x is a path leading to x, and since for each $0 \le x < y \le 1$ we have $r_{[x,y]} \in D_x \cap D_y \cap [(y, (m+1)y - mx) \cup ((m+1)x - my, x)$, the resulting path system $D = \{D_x : x \in [0, 1]\}$ satisfies E.I.C.[m].

Note that inequality (7) is again satisfied when $q_{[x,y]}$ is replaced by $r_{[x,y]}$.

We now must verify that with respect to these path systems D and E, F is path differentiable to f. We only do this for the path system E because a very similar argument works for path system D. Let $x \in [0, 1]$. If $x \notin T$, then $F'(x) = f(x)$, and so we only need concern ourselves with the situation where $x \in T$. We must show that

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y) - F(x)}{y - x} = f(x).$$

To this end, let $\{y_j\}$ be a sequence of points from $E_x \setminus \{x\}$ which converges to x. We shall assume that each $y_j > x$ for specificity. (If a subsequence satisfies $y_j < x$, it can be treated by an analogous argument.) For each j there is a $w_j > x$ such that $y_j = q_{[x,w_j]}$. Furthermore, note that because $q_{[x,w_j]}$ was selected as the first element of the appropriate sequence to satisfy conditions (a) and (b), it will follow that $\{w_j\}$ must also converge to x.

Let $\epsilon > 0$ and choose a natural number $N > 1$ such that $5/2^{N-1} < \epsilon$. Let G denote the union of all the G_τ where

(i) $|\tau| \le N$,
(ii) τ is an extension of $\nu_x|_{|\tau|-1}$,
(iii) $\tau(|\tau|) < \nu_x(|\tau|)$.

Let d denote the positive distance from x to G, and let Δ be the minimum of d and $5/2^{N-1}$. Choose $J \in \mathbb{N}$ so that for all $j > J$, $0 < w_j - x < \Delta$. Let $j > J$. If $w_j \notin T$, then by (b) we have

$$\left| \frac{F(y_j) - F(x)}{y_j - x} - f(x) \right| < |x - w_j| < \epsilon. \tag{9}$$

If $w_j \in T$ and $n_{[x,w_j]} \le N$, then since $w_j \notin G$, we must have $\nu_x(n_{[x,w_j]}) < \nu_{w_j}(n_{[x,w_j]})$; i.e., x is the preferred endpoint of $[x, w_j]$ and thus by (b) we again obtain inequality (9). Finally, if $w_j \in T$ and $n_{[x,w_j]} > N$, there are two possibilities. If x is the preferred endpoint of $[x, w_j]$, then we again obtain inequality (9) from (b). If w_j is the preferred endpoint of $[x, w_j]$, then from

(7) we have

$$\left|\frac{F(y_j) - F(x)}{y_j - x} - f(x)\right| < \frac{5}{2^{n[x,w_j]-1}} < \frac{5}{2^{N-1}} < \epsilon,$$

completing the proof of the theorem. □

As immediate consequences of Theorems 1 and 2 we obtain the following corollaries:

Corollary 1. *Let f be a Baire 1 derivate function of F. Then, F is I.C.-path differentiable to f if and only if (F, f) satisfies Condition $\mathscr{B}$.*

Corollary 2. *Let f be a Baire 1 function which is a bilateral derivate of F. Then, the following are equivalent:*

(1) *(F, f) satisfies Condition $\mathscr{B}$.*
(2) *F is selectively differentiable to f.*
(3) *F is I.I.C.-path differentiable to f.*
(4) *F is one-sided E.I.C.$[m]$-path differentiable to f.*

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