MAyER-VIETORIS FORMULA
FOR THE DETERMINANT OF A LAPLACE OPERATOR
ON AN EVEN-DIMENSIONAL MANIFOLD

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Abstract. Let \( \Delta \) be a Laplace operator acting on differential \( p \)-forms on an even-dimensional manifold \( M \). Let \( \Gamma \) be a submanifold of codimension 1. We show that if \( B \) is a Dirichlet boundary condition and \( R \) is a Dirichlet-Neumann operator on \( \Gamma \), then \( \text{Det}(\Delta + \lambda) = \text{Det}(\Delta + \lambda, B) \text{Det}(\Delta) \) and \( \text{Det}^* \Delta = \frac{1}{\text{Det}(\Delta, B)} \text{Det}(\Delta, B) \text{Det}^* R \). This result was established in 1992 by Burghelea, Friedlander, and Kappeler for a 2-dimensional manifold with \( p = 0 \).

1. Introduction

Let \( M \) be a compact oriented Riemannian manifold of dimension \( d \), and let \( \Gamma \) be a submanifold of \( M \) with dimension \( d - 1 \) such that \( \Gamma \) has a collared neighborhood \( U \) diffeomorphic to \( \Gamma \times (-1, 1) \). Let \( M_\Gamma \) be the compact manifold with boundary \( \Gamma \cup \Gamma \) obtained by cutting \( M \) along \( \Gamma \). Let \( E = \Lambda^p T^* M \) be a \( p \)-th exterior product of the cotangent bundle \( T^* M \), \( i: M_\Gamma \to M \) be the identification map, and \( E_\Gamma := i^* E \).

Define the Dirichlet boundary condition \( (\Delta + \lambda, B) \) to be

\[
(\Delta + \lambda, B): C^\infty(M_\Gamma, E_\Gamma) \to C^\infty(M_\Gamma, E_\Gamma) \oplus C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}),
\]

\[
\omega \mapsto ((\Delta + \lambda)\omega, \omega|_{\partial M_\Gamma}).
\]

Define the Poisson operator \( P_B \) to be the restriction of \( (\Delta + \lambda, B)^{-1} \) to \( 0 \oplus C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}) \). Let \( \nu \) be a unit normal vector field along \( \partial M_\Gamma \); one can extend \( \nu \) to be a global vector field on \( M_\Gamma \) by using a cut-off function. Define the Neumann boundary condition \( C \) to be

\[
C: C^\infty(M_\Gamma, E_\Gamma) \to C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}),
\]

\[
\omega \mapsto \nabla_\nu \omega|_{\partial M_\Gamma}.
\]

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Definition. For any positive real number $\lambda > 0$, define $R(\lambda)$ to be the composition of the following maps:

$$C^\infty(\Gamma, E|_\Gamma) \xrightarrow{\Delta_{ia}} C^\infty(\Gamma, E|_\Gamma) \oplus C^\infty(\Gamma, E|_\Gamma) \oplus 0$$

$$\xrightarrow{p} C^\infty(M_\Gamma, E_\Gamma)$$

$$\xrightarrow{\varphi} C^\infty(\Gamma, E|_\Gamma) \oplus C^\infty(\Gamma, E|_\Gamma)$$

$$\xrightarrow{\Delta_{if}} C^\infty(\Gamma, E|_\Gamma),$$

where $\Delta_{ia}$ is the diagonal inclusion and $\Delta_{if}$ is the difference map.

Then $R(\lambda)$ is a positive definite selfadjoint elliptic operator. When $\lambda = 0$, both the Laplacian $\Delta$ and $R$ have zero eigenvalues and so $\det\Delta = \det R = 0$. In this case we define the modified determinants $\det^* \Delta$ and $\det^* R$ to be the determinants of $\Delta$ and $R$ respectively, when restricted to the orthogonal complement of the null space.

In [BFK], Burghelea, Friedlander, and Kappeler proved that on a 2-dimensional manifold and for the trivial line bundle $E = \Lambda^0 T\ast M$,

1. $\det(\Delta + \lambda) = \det(\Delta + \lambda, B) \det R(\lambda)$ for $\lambda > 0$,
2. $\det^* \Delta = \frac{\sqrt{V}}{l} \det(\Delta, B) \det^* R$,

where $V$ is the area of the manifold and $l$ is the length of $\Gamma$.

Let $\mathcal{H}_p$ be the space of harmonic $p$-forms equipped with the natural inner product $\langle \varphi, \psi \rangle = \int_M \varphi \wedge *\psi = \int_M (\varphi, \psi) d\text{vol}(M)$, where $(\ , \ )$ is a metric in $E = \Lambda^p T\ast M$ induced by the Riemannian metric $g$ on $M$. Let $\mathcal{H}_p|_\Gamma$ be the restriction of harmonic $p$-forms to $\Gamma$. Define an inner product on $\mathcal{H}_p|_\Gamma$ by $\langle \alpha, \beta \rangle_\Gamma = \int_\Gamma (\alpha, \beta) d\mu_\Gamma$, where $d\mu_\Gamma$ is a volume element of $\Gamma$ coming from $g$ restricted to $\Gamma$.

Suppose $k = \text{dim } \mathcal{H}_p$, and let $\psi_1, \ldots, \psi_k$ be an orthonormal basis of $\mathcal{H}_p$ and $\phi_1, \ldots, \phi_k$ be an orthonormal basis of $\mathcal{H}_p|_\Gamma$. Let $J: \mathcal{H}_p \to \mathcal{H}_p|_\Gamma$ denote the restriction map. Let $J(\psi_i) = a_{ij}\phi_j$ and let $A = (a_{ij})_{1 \leq i, j \leq k}$. In this paper we extend the result of Burghelea et al. to arbitrary even dimensions and arbitrary $p$-forms.

If $M$ is a compact oriented Riemannian manifold of dimension $d$ with $d$ even and $E = \Lambda^p T\ast M$, then

Theorem A. $\det(\Delta + \lambda, B) = \det(\Delta + \lambda, B) \det R(\lambda)$ for any $\lambda > 0$.

Theorem B. $\det^* \Delta = \frac{1}{(\det A)^\frac{l}{d}} \det(\Delta, B) \det^* R$.

Remark. If $p = 0$, then $E = M \times R$, and the matrix $A$ is $(\frac{V}{\sqrt{V}})$. Hence Theorem B reduces to

$$\det^* \Delta = \frac{V}{l} \det(\Delta, B) \det^* R,$$

as stated in [BFK].

II. The proof of Theorem A

In [BFK], it is shown that

$$\det(\Delta + \lambda) = c \det(\Delta + \lambda, B) \cdot \det R(\lambda),$$
and that \( \log \det (\Delta + \lambda) \), \( \log \det (\Delta + \lambda, B) \), and \( \log \det R(\lambda) \) admit asymptotic expansions:

\[
\log \det (\Delta + \lambda) \sim \sum_{k=-d}^{\infty} \alpha_k |\lambda|^{-k/2} + \beta_0 \log |\lambda|, \quad \text{with } \alpha_0 = 0,
\]

\[
\log \det R(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j |\lambda|^{-j/2} + \sum_{j=0}^{d} q_j |\lambda|^{j/2} \log |\lambda|, \quad \text{with}
\]

\[
\pi_0 = \sum_{j \in \mathbb{N}} \frac{\partial}{\partial s} \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} J_{d-1}(s, \lambda; x) \varphi_j(x)|_{s=0} \, d\text{vol}(x),
\]

where

\[
J_{d-1}(s, \lambda; x) = \frac{1}{2\pi i} \int_{\mathbb{R}^{d-1}} d\xi \int_{\gamma} \mu^{-s} r_{-1-(d-1)} \left( \mu, \frac{\lambda}{|\lambda|}, x, \xi \right) \, d\mu,
\]

\[
r_{-1} = (\mu - p_1(\lambda, x, \xi))^{-1},
\]

\[
r_{-1-j} = - (\mu - p_1(\lambda, x, \xi))^{-1}
\]

\[
\sigma(R(\lambda)) \sim p_1 + p_0 + p_{-1} + \cdots \quad \text{asymptotic symbol of } R(\lambda), \{\varphi_j\} \quad \text{a partition of unity subordinate to coordinate charts, and } \gamma \quad \text{is a curve on a complex plane enclosing all the eigenvalues of } R(\lambda) \quad \text{counterclockwise.}
\]

Hence

\[
\log c = -\pi_0.
\]

The proof of Theorem A reduces to the verification of the following equation:

\[
p_{1-j}(x, -\xi, \lambda) = (-1)^j p_{1-j}(x, \xi, \lambda).
\]

Then \( r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, -\xi) = (-1)^j r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, \xi) \), so when \( d \) is even, \( r_{-1-(d-1)}(\mu, \frac{\lambda}{|\lambda|}, x, \xi) \) is odd with respect to \( \xi \). So \( J_{d-1} = 0 \) and \( \pi_0 = 0 \). Therefore we conclude

\[
\det(\Delta + \lambda) = \det(\Delta + \lambda, B) \det R(\lambda).
\]

Definition. Let \( U \) be a collared neighborhood of \( \Gamma \) diffeomorphic to \( \Gamma \times (-1, 1) \) with diffeomorphism \( \eta: U \to \Gamma \times (-1, 1) \). Let \( \Gamma_t = \eta^{-1}(\Gamma \times t), -1 < t < 1 \). Let \( N_t^+, N_t^- \) be Neumann operators to each side with respect to \( \Delta + \lambda \); i.e. if \( \varphi \in C_\infty(\Gamma_t, E|_{\Gamma_t}) \), define \( N_t^+(\varphi) = \nabla_{\nu_t} u|_{\Gamma_t} \), where \( (\Delta + \lambda)u = 0 \) in \( M - \Gamma_t, u|_{\Gamma_t} = \varphi \), and \( \nu_t \) is a normal vector field along \( \Gamma_t \).

Then

\[
R(\lambda) = -(N_0^+ + N_0^-).
\]

Lemma 1. In a local coordinate system such that the first fundamental form looks like

\[
\begin{pmatrix}
g_{11}(x, t) & 0 \\ 0 & 1
\end{pmatrix}
\]
on $\Gamma \times (-1, 1)$, the Laplacian is $\Delta = -\frac{d^2}{dt^2} + F(x, t) \frac{d}{dt} + \Delta_t$, where $\Delta_t$ is the Laplacian on $\Gamma_t$ and $F(x, t)$ is a $C^\infty$-function valued $(d \times d)$ matrix. Then

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda),$$
$$\frac{dN_t^-}{dt} = (N_t^-)^2 + F(x, t)N_t^- - (\Delta_t + \lambda).$$

**Remark.** The idea to consider the Neumann operator as a solution of operator-valued differential equations goes back to I. M. Gel’fand.

**Proof.** It is enough to show that the first statement is true. Let $\varphi \in C^\infty(\Gamma_t, E|\Gamma_t)$. Choose $u(x, t) \in C^\infty(M - \Gamma_t, E|\Gamma_t)$ such that $(\Delta + \lambda)u(x, t) = 0$ on $M - \Gamma_t$ and $u(x, t)|\Gamma_t = \varphi$. Then

$$\frac{d}{dt}u(x, t) = N_t^+(u(x, t)),$$
$$\frac{d^2}{dt^2}u(x, t) = \frac{d}{dt}(N_t^+(u(x, t))) = \frac{dN_t^+}{dt}(u(x, t)) + N_t^+ \left( \frac{du}{dt} \right)$$
$$= \left( \frac{dN_t^+}{dt} + (N_t^+)^2 \right) u(x, t),$$

and

$$\frac{d^2}{dt^2}u(x, t) = F(x, t)\frac{du}{dt} + (\Delta_t + \lambda)u(x, t)$$
$$= (F(x, t)N_t^+ + \Delta_t + \lambda)u(x, t).$$

Hence $\frac{dN_t^+}{dt} + (N_t^+)^2 = F(x, t)N_t^+ + (\Delta_t + \lambda)$, so

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda).$$

Let

$$\sigma(N_t^+) \sim \alpha_1 + \alpha_0 + \cdots + \alpha_{1-i} + \cdots,$$
$$\sigma(N_t^-) \sim \beta_1 + \beta_0 + \cdots + \beta_{1-i} + \cdots,$$
$$\sigma(\Delta + \lambda) \sim (\sigma_2 + \lambda) + \sigma_1 + \sigma_0.$$

Note that

$$\sigma_2 + \lambda = \left( \sum_{i,j=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda \right) Id,$$
$$\sigma((N_t^+)^2) \sim \sum_{k=0}^{\infty} \sum_{i,j=1} D_x^\omega \alpha_{1-i} D_x^\omega \alpha_{1-j},$$

where $\omega$ is a multi-index and $D_x = \frac{1}{i} \frac{d}{dx}$.

Since $\frac{dN_t^+}{dt}, \frac{dN_t^-}{dt}$ are first order operators, $-\alpha_1^2 + (\sigma_2 + \lambda) = 0$ and $\beta_1^2 - (\sigma_2 + \lambda) = 0$. So

$$\alpha_1 = \beta_1 = \sqrt{\sum_{i,j=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda} Id$$
and
$$\alpha_1 + \beta_1 = 2 \sqrt{\sum_{i,j=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda} Id.$$
which is even with respect to $\xi$. Note that \( \frac{d\alpha}{dt} = -(2\alpha_0\alpha_1 + d_x\alpha_1 \cdot D_x\alpha_1) + F\alpha_1 + \sigma_1 \) and \( \frac{d\beta}{dt} = (2\beta_0\beta_1 + d_x\beta_1 \cdot D_x\beta_1) + F\beta_1 - \sigma_1 \). Hence

\[
\begin{align*}
\alpha_0 &= \frac{1}{2} \alpha_1^{-1} \left( -\frac{d\alpha_1}{dt} - d_x\alpha_1 \cdot D_x\alpha_1 + F\alpha_1 + \sigma_1 \right), \\
\beta_0 &= \frac{1}{2} \beta_1^{-1} \left( \frac{d\beta_1}{dt} - d_x\beta_1 \cdot D_x\beta_1 - F\beta_1 + \sigma_1 \right).
\end{align*}
\]

Since $\alpha_1 = \beta_1$, it follows that $\alpha_0 + \beta_0 = \alpha_1^{-1}(d_x\alpha_1 \cdot D_x\alpha_1 + \sigma_1)$, which is odd with respect to $\xi$.

**Theorem.** If $\sigma(R(\lambda)) \sim p_1 + p_0 + \cdots + p_{1-j} + \cdots$, then $p_{1-k}$, which is equal to $-\alpha_{1-k} - \beta_{1-k}$, is even (odd) with respect to $\xi$ when $k$ is even (odd).

**Proof.** Note that one has

\[
\begin{align*}
\alpha_{1-k} &= \frac{1}{2} \alpha_1^{-1} \left\{ -\frac{d\alpha_1}{dt} - \sum_{i+j+|\omega|=k} \frac{1}{\omega!} d^\omega_x \alpha_1 - i D^\omega_x \alpha_{1-j} + F\alpha_{1-(k-1)} \right\}, \\
\beta_{1-k} &= \frac{1}{2} \beta_1^{-1} \left\{ \frac{d\beta_1}{dt} - \sum_{i+j+|\omega|=k} \frac{1}{\omega!} d^\omega_x \beta_1 - i D^\omega_x \beta_{1-k} - F\beta_{1-(k-1)} \right\}.
\end{align*}
\]

Since $\alpha_1 = \beta_1 = \sqrt{\sum_{i,j} g_{ij}\xi_i \xi_j + \lambda I}$, we can use (*) for each $\alpha_{1-i}$, $\beta_{1-j}$ to express $\alpha_{1-k}$ and $\beta_{1-k}$ in terms of $\alpha_1$, $\sigma_1$, and $\sigma_0$. In fact,

\[
\begin{align*}
\alpha_{1-k} &= \sum_r (-1)^r \frac{1}{2} \alpha_1^{-1} \frac{d}{dt} \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d}{dt} \cdots \frac{1}{2} \alpha_1^{-1} \left( \frac{d}{dt} q_r^{k-r} \right) \right. \right. \\
&\quad \left. \left. \left. \cdots \right\} \right\} \\
&\quad + \sum_s (-1)^s \frac{1}{2} \alpha_1^{-1} F \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ F \cdots \frac{1}{2} \alpha_1^{-1} (F q_s^{k-s}) \right. \right. \right. \\
&\quad \left. \left. \left. \cdots \right\} \right\} + P_k
\end{align*}
\]

and

\[
\begin{align*}
\beta_{1-k} &= \sum_r (-1)^r \frac{1}{2} \alpha_1^{-1} \frac{d}{dt} \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d}{dt} \cdots \frac{1}{2} \alpha_1^{-1} \left( \frac{d}{dt} q_r^{k-r} \right) \right. \right. \\
&\quad \left. \left. \left. \cdots \right\} \right\} \\
&\quad + \sum_s (-1)^s \frac{1}{2} \alpha_1^{-1} F \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ F \cdots \frac{1}{2} \alpha_1^{-1} (F q_s^{k-s}) \right. \right. \right. \\
&\quad \left. \left. \left. \cdots \right\} \right\} + P_k,
\end{align*}
\]

where $\frac{d}{dt}$ appears $r$ times and $F$ appears $s$ times, respectively, and $q_r^{k-r}$, $q_s^{k-s}$, $P_k$ are functions consisting of some jets of $\alpha_1$, $\alpha_1^{-1}$, $\sigma_1$, and $\sigma_0$ satisfying

\[
\begin{align*}
q_r^{k-r}(x, -\xi) &= (-1)^{k-r} q_r^{k-r}(x, \xi), \\
q_s^{k-s}(x, -\xi) &= (-1)^{k-s} q_s^{k-s}(x, \xi), \\
P_k(x, -\xi) &= (-1)^k P_k(x, \xi).
\end{align*}
\]

Hence

\[
-p_{1-k} = \alpha_{1-k} + \beta_{1-k}
= 2 \sum_{r: \text{even}} \frac{1}{2} \alpha_1^{-1} \frac{d}{dt} \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d}{dt} \cdots \frac{1}{2} \alpha_1^{-1} \left( \frac{d}{dt} q_r^{k-r} \right) \right. \right. \\
&\quad \left. \left. \cdots \right\} \right\} \\
+ 2 \sum_{s: \text{even}} \frac{1}{2} \alpha_1^{-1} F \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ F \cdots \frac{1}{2} \alpha_1^{-1} (F q_s^{k-s}) \right. \right. \right. \\
&\quad \left. \left. \left. \cdots \right\} \right\} + 2P_k.
\]
and so \( p_{1-k} \) is even if \( k \) is even, and \( p_{1-k} \) is odd if \( k \) is odd, with respect to \( \xi \).

### III. The proof of Theorem B

**Lemma 2.** \( R(\varepsilon)^{-1} = J \circ (\Delta + \varepsilon)^{-1} \circ (\varepsilon \otimes \delta_{\Gamma}) \), where \( J \) is the restriction map to \( \Gamma \) and \( \delta_{\Gamma} \) is the Dirac \( \delta \)-function along \( \Gamma \).

**Proof.** For \( \varphi \in C^\infty(\Gamma, \mathcal{E}|_{\Gamma}) \) choose \( u \) such that \((\Delta + \varepsilon)u = 0\) in \( M - \Gamma \) and \( u|_{\Gamma} = \varphi \). Then

\[
\frac{du}{dt} = \begin{cases} \nabla_{\nu_i} u = N_i^+(u(x,t)) & \text{for } t > 0, \\ -\nabla_{-\nu_i} u = -N_i^-(u(x,t)) & \text{for } t < 0. \end{cases}
\]

Now \( R(\varepsilon)\varphi = -N_0^+(\varphi) - N_0^-(\varphi) \). So

\[
\frac{du}{dt} = \begin{cases} -R(\varepsilon)\varphi + N_i^+(u(x,t)) + R(\varepsilon)\varphi, & t \geq 0, \\ -N_i^- (u(x,t)), & t < 0. \end{cases}
\]

Let

\[
v(x, t) = \begin{cases} N_i^+(u(x,t)) + R(\varepsilon)\varphi, & t \geq 0, \\ -N_i^- (u(x,t)), & t < 0. \end{cases}
\]

Then

\[
\frac{dv}{dt}(x, t) = \frac{d}{dt} N_i^+(u(x,t)) = \left( \frac{dN_i^+}{dt} + (N_i^+)^2 \right) u(x,t)
\]

by Lemma 1. In the same way for \( t < 0 \), \( \frac{dv}{dt} = -(F(x,t)N_i^- + \Delta_t + \varepsilon)u(x,t) \).

Hence

\[
\frac{d^2u}{dt^2} = -R(\varphi) \otimes \delta_{\Gamma} + \frac{dv}{dt}(x, t)
\]

\[
= -R(\varphi) \otimes \delta_{\Gamma} + (F(x,t)N_i^+ + \Delta_t + \varepsilon)u(x,t),
\]

\[
-\frac{d^2u}{dt^2} + (F(x,t)N_i^+ + \Delta_t + \varepsilon)u(x,t) = R(\varphi) \otimes \delta_{\Gamma},
\]

\[
(\Delta + \varepsilon)u = R(\varphi) \otimes \delta_{\Gamma}.
\]

Hence

\[
R(\varepsilon)^{-1}(\varphi) = J \circ (\Delta + \varepsilon)^{-1} \circ (\varphi \otimes \delta_{\Gamma}).
\]

**Theorem B.** \( \text{Det}^*(\Delta) = \frac{1}{(\text{det} A)^k} \text{Det}(\Delta, B) \cdot \text{Det}^* R \).

**Proof.** Let \( k = \dim \mathcal{F} \). Then

(1) \[
\log \text{Det}(\Delta + \varepsilon) = k \log \varepsilon + \log \text{Det}^*(\Delta) + o(\varepsilon).
\]

Denote by \( \mu_j = \mu_j(\varepsilon) \) \((j \geq 1)\) the eigenvalues of \( R(\varepsilon) \) with \( 0 < \mu_1(\varepsilon) \leq \cdots \leq \mu_k(\varepsilon) < \mu_{k+1}(\varepsilon) \leq \cdots \). It is clear that \( \lim_{\varepsilon \to 0} \mu_j(\varepsilon) = 0 \) for \( 1 \leq j \leq k \). Then

\[
\log \text{Det} R(\varepsilon) = \log \mu_1(\varepsilon) \cdots \mu_k(\varepsilon) + \log \text{Det}^* R + o(\varepsilon).
\]
Now we want to calculate $\mu_1(e) \cdots \mu_k(e)$. Let $\{\psi_j\}_{j \geq 1}$ be the complete orthonormal system of eigenforms of $\Delta$ with eigenvalue $\lambda_j$ in $L^2(M, E)$. For any $\varphi \in C^\infty(\Gamma, E|_\Gamma)$, $\varphi \otimes \delta_\Gamma \in H^{-1}(M, E)$ and $(\Delta + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma) \in L^2(M, E)$.

$$(\Delta + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma), \psi_j = (\varphi \otimes \delta_\Gamma, (\Delta + \varepsilon)^{-1}\psi_j) = (\varphi \otimes \delta_\Gamma, \frac{1}{\lambda_j + \varepsilon}\psi_j)$$

$$= \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma,$$

where $d\mu_\Gamma$ is a volume element in $\Gamma$. Hence

$$(\Delta + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma) = \sum_{j=1}^\infty \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j.$$ 

Let $\psi_1, \ldots, \psi_k$ be harmonic forms and $\lambda_1 = \cdots = \lambda_k = 0$. Then

$$(2) \quad R(e)^{-1}\varphi = \frac{1}{e} \sum_{i=1}^k \int_\Gamma (\varphi, \psi_i) d\mu_\Gamma \cdot \psi_i|_\Gamma + \sum_{j=k+1}^\infty \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j|_\Gamma.$$ 

From (2), one can check that $R(e)^{-1}$ is symmetric and positive definite; it follows that $R(e)$ is also symmetric and positive definite.

Let $\phi_1(e), \ldots, \phi_k(e)$ be orthonormal eigenforms of $R(e)$ corresponding to eigenvalues $\mu_1(e), \ldots, \mu_k(e)$. Then $\phi_j(e) \to \psi_j$ as $e \to 0$, where $\psi_j$ is the restriction of a harmonic form to $\Gamma$ with $\langle \phi_j, \phi_j \rangle_\Gamma = 1$. Let $a_{ij}(e) = \langle \psi_i, \phi_j(e) \rangle_\Gamma$, $1 \leq i, j \leq k$, and $A(e) = (a_{ij}(e))$. Now $\psi_i|_\Gamma = a_{ij}(e)\phi_j(e) + \psi_i(e)|_\Gamma$ for some $\psi_i(e)|\Gamma \in (\text{span}\{\phi_1(e), \ldots, \phi_k(e)\})^\perp$. Define

$$I: C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma)$$

by

$$\varphi \mapsto \sum_{j=1}^k \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j|_\Gamma = \sum_{j=1}^k (\varphi, \psi_j) \Gamma \cdot \psi_j|_\Gamma.$$ 

Then

$$\langle I(\phi_i(e)), \phi_j(e) \rangle_\Gamma = \sum_{l=1}^k a_{il}(e)a_{lj}(e) = (A^A)_{ij}(e).$$ 

Define

$$G_e: C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma)$$

by

$$\varphi \mapsto \sum_{j=k+1}^\infty \frac{1}{\lambda_j + \varepsilon} (\varphi, \psi_j) \Gamma \cdot \psi_j|_\Gamma.$$ 

Then $\|G_e\|_{L^2}$ converges to $\frac{1}{\lambda_{k+1}} > 0$ as $e \to 0$. Now

$$R(e)^{-1}(\varphi) = \frac{1}{e} I(\varphi) + G_e(\varphi).$$
For \( 1 \leq j \leq k \),
\[
\frac{1}{\mu_j(\varepsilon)} = \langle R(\varepsilon)^{-1} \phi_j(\varepsilon), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle I(\phi_j(\varepsilon)), \phi_j(\varepsilon) \rangle + \langle G_\varepsilon(\phi_j(\varepsilon)), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle (\varepsilon \varepsilon)_{jj}(\varepsilon) \rangle + N_j(\varepsilon),
\]
where \( N_j(\varepsilon) = \langle G_\varepsilon(\phi_j(\varepsilon)), \phi_j(\varepsilon) \rangle \) is bounded as \( \varepsilon \to 0 \). For \( i \neq j \), \( 1 \leq i, j \leq k \),
\[
0 = \langle R(\varepsilon)^{-1} \phi_i(\varepsilon), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle I(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle + \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle (\varepsilon \varepsilon)_{ij}(\varepsilon) \rangle + \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle.
\]
Since \( (\varepsilon \varepsilon)_{ij}(\varepsilon) \) and \( \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle \) are bounded, \( (\varepsilon \varepsilon)_{ij}(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). So
\[
\frac{1}{\mu_1(\varepsilon) \cdots \mu_k(\varepsilon)} = \left( \frac{1}{\varepsilon} \langle (\varepsilon \varepsilon)_{11}(\varepsilon) \rangle + N_1(\varepsilon) \right) \cdots \left( \frac{1}{\varepsilon} \langle (\varepsilon \varepsilon)_{kk}(\varepsilon) \rangle + N_k(\varepsilon) \right) \\
= \frac{1}{\varepsilon^k (\text{det } A)^2} \left( \frac{(\varepsilon \varepsilon)_{11}(\varepsilon \varepsilon)_{22} \cdots (\varepsilon \varepsilon)_{kk}(\varepsilon)}{(\text{det } A)^2} + \varepsilon \cdot \tilde{N}(\varepsilon) \right),
\]
where \( \tilde{N}(\varepsilon) \) is bounded as \( \varepsilon \to 0 \). Hence
\[
(3) \quad \log \text{Det } R(\varepsilon) = k \log \varepsilon - \log(\text{det } A)^2 + \log \text{Det}^* R + o(\varepsilon).
\]
If we combine equation (1) and equation (3), we get
\[
\log \text{Det}^* \Delta = -\log(\text{det } A)^2 + \log \text{Det}^* R + \log \text{Det}(\Delta, B).
\]

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REFERENCES