TRIANGULAR TRUNCATION AND NORMAL LIMITS
OF NILPOTENT OPERATORS

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Dedicated to the memory of Domingo Herrero, who dazzled us with his brilliance,
charmed us with his wit, and warmed us with his heart

Abstract. We show that, as \( n \to \infty \), the product of the norm of the triangular
truncation map on the \( n \times n \) complex matrices with the distance from the norm-
one hermitian \( n \times n \) matrices to the nilpotents converges to \( 1/2 \). We also include
an elementary proof of D. Herrero's characterization of the normal operators
that are norm limits of nilpotents.

Suppose \( n \) is a positive integer and let \( \mathcal{M}_n \), \( \mathcal{T}_n \), \( \mathcal{N}_n \) denote, respectively, the
sets of all \( n \times n \) complex matrices, strictly upper triangular \( n \times n \) matrices, and
nilpotent \( n \times n \) matrices. There is a natural mapping \( \tau_n : \mathcal{M}_n \to \mathcal{T}_n \), namely,
\( \tau_n(T) \) replaces the entries on or below the main diagonal of \( T \) with zeroes.
The map \( \tau_n \) is called triangular truncation on \( \mathcal{M}_n \).

On an infinite-dimensional space, the triangular truncation mapping does not
always yield the matrix of a bounded operator. This is related to the fact that the
range of the mapping that sends a bounded harmonic function on the unit disk
to its analytic part is not included in \( H^\infty \). For example, if \( f(z) = \log(1 - z) \),
then \( u = 2i \text{Im}(f) \) is bounded in modulus by \( \pi \), but the analytic part of \( u \),
namely \( f \), is not bounded. In terms of Toeplitz operators, \( T_u \) is an operator
with norm \( \pi \), but the upper triangular truncation of \( T_u \) is the formal matrix
for \( T_f \), which is not a bounded operator. The matrix for \( T_u \) is the matrix
whose \((i, i)\)-entry is 0 and \((i, j)\)-entry is \( 1/(j - i) \) for \( 1 \leq i \neq j < \infty \).
For each positive integer \( n \), let \( T_{u, n} \) be the \( n \times n \) upper-left-hand corner of \( T_u \),
i.e., the \((i, i)\)-entry of \( T_{u, n} \) is 0, and the \((i, j)\)-entry of \( T_{u, n} \) is \( 1/(j - i) \) for
\( 1 \leq i \neq j \leq n \). It follows that \( \|T_{u, n}\| \leq \pi \) for each \( n \), and that \( \|	au_n(T_{u, n})\| \to \infty \)
as \( n \to \infty \). Hence \( \|\tau_n\| \to \infty \) as \( n \to \infty \).

Much work has been done in determining \( \|\tau_n\| \). S. Kwapien and A. Pelczynski
[KP, pp. 45–48] proved in 1970 that \( \|\tau_n\| = O(\log(n)) \), K. Davidson [D,
p. 39] proved that \( \frac{4}{2n} \leq \liminf_{n \to \infty} \|\tau_n\|/\log(n) \), and, in 1993, J. R. Angelos,
C. Cowen, and S. K. Narayan [ACN] proved that

\[
\lim_{n \to \infty} \|\tau_n\|/\log(n) = 1/\pi.
\]

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All three of the papers cited above made use of the matrices $T_{u,n}$. It was shown in [ACN] that $\|T_{u,n}\| \leq \pi$ and, for $n \geq 2$, $\|\tau_n(T_{u,n})\| \geq \log(n) - 1$, which implies that $\|\tau_n\| \geq \frac{\log(n) - 1}{\pi}$.

For each positive integer $n$ we define a number $\delta_n$ that measures how closely nilpotent $n \times n$ matrices can approximate norm-one hermitian ones, namely,

$$\delta_n = \inf\{\|T - N\| : T \in \mathcal{M}_n, N \in \mathcal{M}_n, T = T^*, \|T\| = 1\}.$$

It follows from the work of D. Herrero [H1] that $\delta_n \to 0$ as $n \to \infty$, and it was shown in [AFHV, Corollary A1.12] that

$$\delta_n \geq \frac{\log(2)}{(1.038) \log(2) + \log(n)}.$$

It was also pointed out in [AFHV] that W. Kahan [K] proved that

$$\delta_n \leq \frac{\pi + \log(2)}{2 \log(n)}.$$

In this note we show a direct connection between $\delta_n$ and $\tau_n$ and prove $\lim_{n \to \infty} \delta_n \|\tau_n\| = 1/2$. We then show how the fact that $\delta_n \to 0$ can be used to give an elementary proof of D. Herrero's theorem [H1] (Theorem 2 below), characterizing normal limits of nilpotent operators.

**Theorem 1.** $\lim_{n \to \infty} \delta_n \|\tau_n\| = 1/2$.

**Proof.** Let $N_n = \tau_n(T_{u,n})$. Then $\|N_n\| \geq \log(n) - 1$ and $\|\text{Im}(N_n)\| = \| - iT_{u,n}/2 \| \leq \pi/2$.

Hence, $\|\text{Re}(N_n)\| \geq \|N_n\| - \|\text{Im}(N_n)\| \geq \log(n) - 1 - \pi/2$. Thus,

$$\delta_n \leq \|N_n - \text{Re}(N_n)\|/\|\text{Re}(N_n)\| \leq \frac{\pi/2}{\log(n) - 1 - \pi/2}.$$

Hence

$$\limsup_{n \to \infty} \delta_n \|\tau_n\| \leq \limsup_{n \to \infty} \frac{\|\tau_n\| (\pi/2) \log(n)}{\log(n) \log(n) - 1 - \pi/2} = \frac{1 \pi}{\pi 2} = \frac{1}{2}.$$

On the other hand, suppose $A \in \mathcal{M}_n$, $N \in \mathcal{M}_n$, $A = A^*$, $\|A\| = 1$, and $\|A - N\| = \delta_n$. Then $\|N\| \geq 1 - \delta_n$, $\|\text{Im}(N)\| = \|\text{Im}(A - N)\| \leq \delta_n$. Since $N \in \mathcal{M}_n$, we can assume, via unitary equivalence that $N \in \mathcal{F}_n$. Hence

$$\tau_n(\text{Im}(N)) = N/2.$$

Hence,

$$\|\tau_n\| \geq \|N/2\|/\|\text{Im}(N)\| \geq \frac{(1 - \delta_n)/2}{\delta_n},$$

which implies that $\liminf_{n \to \infty} \delta_n \|\tau_n\| \geq \liminf_{n \to \infty} (1 - \delta_n)/2 = 1/2$. □

**Corollary.** $\lim_{n \to \infty} \delta_n \log(n) = \pi/2$.

We now show that $\lim_{n \to \infty} \delta_n = 0$ leads to an elementary proof of Herrero's theorem on normal limits of nilpotent operators. Here $\sigma(T)$ denotes the spectrum of $T$.  

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Theorem 2 (Herrero [H1]). Suppose \( T \) is a normal operator on a separable, infinite-dimensional Hilbert space, \( \sigma(T) \) is connected, and \( 0 \in \sigma(T) \). Then \( T \) is a norm-limit of nilpotent operators.

The converse of the above theorem is also true. If \( T \) is any operator that is a limit of noninvertible operators, then \( 0 \in \sigma(T) \), since the set of invertible operators is open. The fact that a norm limit of nilpotent operators must have connected spectrum uses the Riesz-Dunford functional calculus, namely, the first result in D. Herrero’s book [H2] on approximation.

Proposition 3 [H2, Theorem 1.1]. Suppose \( A \) is an operator and \( \sigma(A) \) is the disjoint union of compact subsets \( \sigma_0 \) and \( \sigma_1 \), with \( \sigma_1 \) nonempty. Suppose also that \( \Omega \) is a bounded open set containing \( \sigma_1 \) such that the closure \( \Omega^- \) is disjoint from \( \sigma_0 \). If \( B \) is any operator with \( \|B - A\| < \inf\{||(z - A)^{-1}\|^{-1} : z \in \partial \Omega\} \), then \( \sigma(B) \cap \Omega \neq \emptyset \).

Proof of Theorem 2. From the fact that \( \delta_n \to 0 \), we know that there is a sequence \( \{A_n\} \) of norm-one hermitian matrices and a sequence \( \{N_n\} \) of nilpotent matrices such that \( \|A_n - N_n\| \to 0 \). It follows that \( \|N_n\| \to 1 \), and we conclude that \( \|A_n^2 - N_n^2\| \to 0 \). Since \( \|A_n^2\| = 1 \) and \( N_n^2 \) is nilpotent, we can assume without loss of generality that \( 0 \leq A_n \leq 1 \).

We remarked above that a norm limit of nilpotent operators must have connected spectrum. The \( A_n \)'s, being finite matrices, cannot have connected spectrum; however, their spectra must try to fill out the interval \([0, 1]\) as \( n \to \infty \).

Lemma 4. For each \( t \) in \([0, 1]\), \( \text{dist}(t, \sigma(A_n)) \leq \|A_n - N_n\| \).

Proof. Suppose \( 0 < t < 1 \), and assume that \( t \notin \sigma(A_n) \). If we apply Proposition 3 with \( A = A_n \), \( B = N_n \), and \( \Omega \) the disk centered at 1 with radius \( 1 - t \), then the fact that \( \sigma(N_n) \cap \Omega = \emptyset \) and the fact that \( \inf\{||(z - A)^{-1}\|^{-1} : z \in \partial \Omega\} = \text{dist}(t, \sigma(A_n)) \) implies the desired inequality. \( \square \)

We next show how the operator \( M_x \), multiplication by the independent variable \( x \), on the space \( L^2[0, 1] \) with Lebesgue measure is a norm limit of nilpotents. Later, we will modify the argument to handle the general case.

Lemma 5. The operator \( M_x \) is a norm limit of nilpotent operators.

Proof. Let \( 0 \leq t_{n_1} \leq t_{n_2} \leq \cdots \leq t_{n_{k(n)}} = 1 \) be the distinct eigenvalues of \( A_n \). Let \( A_n^{(\infty)} \), \( N_n^{(\infty)} \) denote, respectively, a direct sum of infinitely many copies of \( A_n \), \( N_n \). Define \( f_n : [0, 1] \to [0, 1] \) to be the simple function taking the value \( t_{nk} \) on the interval \([t_{nk-1}, t_{nk}) \) (where \( t_{n0} = 0 \)).

It follows from Lemma 4 that \( \|x - f_n(x)\|_\infty \to 0 \); whence \( \|M_{f_n} - M_x\| \to 0 \). However, \( A_n^{(\infty)} \) and \( M_{f_n} \) are both diagonalizable operators with eigenvalues \( t_{n_1}, t_{n_2}, \ldots, t_{n_{k(n)}} \), each having infinite multiplicity. Thus there is a unitary operator \( U_n \) such that \( U_n^{*} A_n^{(\infty)} U_n = M_{f_n} \), for each \( n \geq 1 \). Hence

\[
\|U_n^{*} N_n^{(\infty)} U_n - M_x\| \leq \|U_n^{*} [N_n^{(\infty)} - A_n^{(\infty)}] U_n\| + \|U_n^{*} A_n^{(\infty)} U_n - M_{f_n}\| + \|M_{f_n} - M_x\|
\]

\[
= \|N_n - A_n\| + \|M_{f_n} - M_x\| \to 0 \quad \text{as} \quad n \to \infty.
\]
Since $U_n^*N_n^{(∞)}U_n$ is a nilpotent operator for every $n$, we have proved that $M_x$ is a norm limit of nilpotent operators. □

Lemma 6. If $T$ is a normal operator, $0 \in \sigma(T)$, and $\sigma(T)$ is connected, then there is a sequence \{$p_n$\} of polynomials vanishing at $0$ and a sequence \{$W_n$\} of unitary operators with $\|W_n^*p_n(M_x)W_n - T\| \to 0$.

Proof. Suppose $\varepsilon > 0$, and let $\Omega = \{z + w: z \in \sigma(T), |w| < \varepsilon\}$. Then $\Omega$ is open and connected; whence $\Omega$ is path-connected. Since $\sigma(T)$ is compact, we can find a finite subset $\mathcal{F} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ of $\sigma(T)$ with $\lambda_1 = 0$ such that every point in $\sigma(T)$ is within $\varepsilon$ of some point in $\mathcal{F}$. It follows from the spectral theorem that there is a diagonalizable operator $D$ whose set of eigenvalues is $\mathcal{F}$, with each eigenvalue having infinite multiplicity, such that $\|T - D\| < \varepsilon$. Since $\Omega$ is path-connected, there is a continuous function $\psi: [0, 1] \to \Omega$ such that $\psi(0) = 0$ and the range of $\psi$ includes $\mathcal{F}$. Since $\psi$ is uniformly continuous on $[0, 1]$, there is a positive integer $m$ such that $\|\psi - h\|_{\infty} < \varepsilon$ where $h$ is the simple function taking on the value $\psi(k/m)$ on the interval $[k/m, (k + 1)/m)$, for $0 \leq k \leq m - 1$. Hence

$$\|\psi(M_x) - h(M_x)\| = \|M_\psi - M_h\| < \varepsilon.$$

However, $M_h$ is a diagonalizable operator, with each eigenvalue having infinite multiplicity, and such that every point in $\mathcal{F}$ is within $\varepsilon$ of the set of eigenvalues $M_h$.

It follows that every eigenvalue of $D$ is within $\varepsilon$ of an eigenvalue of $M_h$, and that each eigenvalue of $M_h$ is within $2\varepsilon$ of an eigenvalue of $D$. By rearranging eigenvalues, we can find a unitary operator $W$ such that $\|W^*M_hW - D\| < 2\varepsilon$. Hence $\|W^*\psi(M)xW - D\| < 3\varepsilon$. Since, by the Weierstrass approximation theorem, $\psi$ is a uniform limit of polynomials on $[0, 1]$ (that vanish at 0 since $\psi$ vanishes at 0), there is a polynomial $p$ with $p(0) = 0$ such that $\|p - \psi\|_{\infty} < \varepsilon$. Hence,

$$\|W^*p(M_x)W - T\| \leq \|W^*[p(M_x) - \psi(M_x)]W\| + \|W^*\psi(M_x)W - D\| + \|D - T\|$$

$$< \varepsilon + 3\varepsilon + \varepsilon = 5\varepsilon.$$

This completes the proof of the lemma. □

The proof of Theorem 2 is completed by noting that, since the set $\mathcal{N}$ of nilpotent operators is closed under unitary equivalence and under evaluation of polynomials that vanish at 0, the same is true for the norm closure $\mathcal{N}^-$. Since $M_x \in \mathcal{N}^-$, it follows that, for every unitary operator $W$ and for every polynomial $p$ with $p(0) = 0$, we have $W^*p(M_x)W \in \mathcal{N}^-$. The preceding lemma clearly implies that $\mathcal{N}^-$ contains every normal operator whose spectrum is connected and contains 0. This completes the proof of Theorem 2.

It follows from the proof of Lemma 1 that we could take

$$N_n = [\pi\tau_n(T_{u,n})/\log(n)]^2 \quad \text{and} \quad A_n = [\Re(N_n)]^2$$

in the proof of Theorem 2. This, modulo the problem of computing the eigenvectors and eigenvalues of $A_n$, gives a “concrete” construction of a sequence of nilpotents that converge in norm to $M_x$ on $L^2[0, 1]$.

The description of the closure of the set of all nilpotent operators on a separable infinite-dimensional Hilbert space is a deep theorem of C. Apostol,
C. Foias, and D. Voiculescu [AFV]. A formula for the distance from an operator to the set of nilpotent operators is contained in [AFHV, 12.7].

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