PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

TRIANGULAR TRUNCATION AND NORMAL LIMITS OF NILPOTENT OPERATORS

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(Communicated by Palle E. T. Jorgensen)

Dedicated to the memory of Domingo Herrero, who dazzled us with his brilliance, charmed us with his wit, and warmed us with his heart

ABSTRACT. We show that, as $n \to \infty$, the product of the norm of the triangular truncation map on the $n \times n$ complex matrices with the distance from the norm-one hermitian $n \times n$ matrices to the nilpotents converges to $1/2$. We also include an elementary proof of D. Herrero's characterization of the normal operators that are norm limits of nilpotents.

Suppose n is a positive integer and let $\mathscr{M}_n$, $\mathscr{T}_n$, $\mathscr{N}_n$ denote, respectively, the sets of all $n \times n$ complex matrices, strictly upper triangular $n \times n$ matrices, and nilpotent $n \times n$ matrices. There is a natural mapping $\tau_n : \mathscr{M}_n \to \mathscr{T}_n$, namely, $\tau_n(T)$ replaces the entries on or below the main diagonal of T with zeroes. The map τ_n is called *triangular truncation* on $\mathscr{M}_n$.

On an infinite-dimensional space, the triangular truncation mapping does not always yield the matrix of a bounded operator. This is related to the fact that the range of the mapping that sends a bounded harmonic function on the unit disk to its analytic part is not included in H^∞. For example, if $f(z) = \log(1 - z)$, then $u = 2i\,\mathrm{Im}(f)$ is bounded in modulus by π, but the analytic part of u, namely f, is not bounded. In terms of Toeplitz operators, T_u is an operator with norm π, but the upper triangular truncation of T_u is the formal matrix for T_f, which is not a bounded operator. The matrix for T_u is the matrix whose (i, i)-entry is 0 and (i, j)-entry is $1/(j - i)$ for $1 \le i \ne j < \infty$. For each positive integer n, let $T_{u,n}$ be the $n \times n$ upper-left-hand corner of T_u, i.e., the (i, i)-entry of $T_{u,n}$ is 0, and the (i, j)-entry of $T_{u,n}$ is $1/(j - i)$ for $1 \le i \ne j \le n$. It follows that $\|T_{u,n}\| \le \pi$ for each n, and that $\|\tau_n(T_{u,n})\| \to \infty$ as $n \to \infty$. Hence $\|\tau_n\| \to \infty$ as $n \to \infty$.

Much work has been done in determining $\|\tau_n\|$. S. Kwapien and A. Pelczynski [KP, pp. 45–48] proved in 1970 that $\|\tau_n\| = O(\log(n))$, K. Davidson [D, p. 39] proved that $\frac{4}{5\pi} \le \liminf_{n\to\infty} \|\tau_n\|/\log(n)$, and, in 1993, J. R. Angelos, C. Cowen, and S. K. Narayan [ACN] proved that

$$\lim_{n\to\infty} \|\tau_n\|/\log(n) = 1/\pi.$$

Received by the editors September 22, 1993; this paper was presented at a miniconference in honor of Eric Nordgren's sixtieth birthday held at the University of New Hampshire in June, 1993.
1991 *Mathematics Subject Classification.* Primary 47A58, 47B15; Secondary 15A60.

0002-9939/95 \$1.00 + \$.25 per page

All three of the papers cited above made use of the matrices $T_{u,n}$. It was shown in [ACN] that $\|T_{u,n}\| \leq \pi$ and, for $n \geq 2$, $\|\tau_n(T_{u,n})\| \geq \log(n) - 1$, which implies that $\|\tau_n\| \geq \frac{\log(n)-1}{\pi}$.

For each positive integer n we define a number δ_n that measures how closely nilpotent $n \times n$ matrices can approximate norm-one hermitian ones, namely,

$$\delta_n = \inf\{\|T - N\|: T \in \mathcal{M}_n, N \in \mathcal{N}_n, T = T^*, \|T\| = 1\}.$$

It follows from the work of D. Herrero [H1] that $\delta_n \to 0$ as $n \to \infty$, and it was shown in [AFHV, Corollary A1.12] that

$$\delta_n \geq \frac{\log(2)}{(1.038)\log(2) + \log(n)}.$$

It was also pointed out in [AFHV] that W. Kahan [K] proved that

$$\delta_n \leq \frac{\pi + \log(2)}{2\log(n)}.$$

In this note we show a direct connection between δ_n and τ_n and prove $\lim_{n\to\infty} \delta_n \|\tau_n\| = 1/2$. We then show how the fact that $\delta_n \to 0$ can be used to give an elementary proof of D. Herrero's theorem [H1] (Theorem 2 below), characterizing normal limits of nilpotent operators.

Theorem 1. $\lim_{n\to\infty} \delta_n \|\tau_n\| = 1/2$.

Proof. Let $N_n = \tau_n(T_{u,n})$. Then

$$\|N_n\| \geq \log(n) - 1 \quad \text{and} \quad \|\operatorname{Im}(N_n)\| = \|-iT_{u,n}/2\| \leq \pi/2.$$

Hence, $\|\operatorname{Re}(N_n)\| \geq \|N_n\| - \|\operatorname{Im}(N_n)\| \geq \log(n) - 1 - \pi/2$. Thus,

$$\delta_n \leq \|(N_n - \operatorname{Re}(N_n))/\|\operatorname{Re}(N_n)\|\,\| \leq \frac{\pi/2}{\log(n) - 1 - \pi/2}.$$

Hence

$$\limsup_{n\to\infty} \delta_n \|\tau_n\| \leq \limsup_{n\to\infty} \frac{\|\tau_n\|}{\log(n)} \frac{(\pi/2)\log(n)}{\log(n) - 1 - \pi/2} = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}.$$

On the other hand, suppose $A \in \mathcal{M}_n$, $N \in \mathcal{N}_n$, $A = A^*$, $\|A\| = 1$, and $\|A - N\| = \delta_n$. Then $\|N\| \geq 1 - \delta_n$, $\|\operatorname{Im}(N)\| = \|\operatorname{Im}(A - N)\| \leq \delta_n$. Since $N \in \mathcal{N}_n$, we can assume, via unitary equivalence that $N \in \mathcal{T}_n$. Hence $\tau_n(\operatorname{Im}(N)) = N/2$.

Hence,

$$\|\tau_n\| \geq \|N/2\|/\|\operatorname{Im}(N)\| \geq \frac{(1-\delta_n)/2}{\delta_n},$$

which implies that $\liminf_{n\to\infty} \delta_n \|\tau_n\| \geq \liminf_{n\to\infty} (1 - \delta_n)/2 = 1/2$. □

Corollary. $\lim_{n\to\infty} \delta_n \log(n) = \pi/2$.

We now show that $\lim_{n\to\infty} \delta_n = 0$ leads to an elementary proof of Herrero's theorem on normal limits of nilpotent operators. Here $\sigma(T)$ denotes the spectrum of T.

Theorem 2 (Herrero [H1]). *Suppose T is a normal operator on a separable, infinite-dimensional Hilbert space, $\sigma(T)$ is connected, and $0 \in \sigma(T)$. Then T is a norm-limit of nilpotent operators.*

The converse of the above theorem is also true. If T is any operator that is a limit of noninvertible operators, then $0 \in \sigma(T)$, since the set of invertible operators is open. The fact that a norm limit of nilpotent operators must have connected spectrum uses the Riesz-Dunford functional calculus, namely, the first result in D. Herrero's book [H2] on approximation.

Proposition 3 [H2, Theorem 1.1]. *Suppose A is an operator and $\sigma(A)$ is the disjoint union of compact subsets σ_0 and σ_1, with σ_1 nonempty. Suppose also that Ω is a bounded open set containing σ_1 such that the closure Ω^- is disjoint from σ_0. If B is any operator with $\|B - A\| < \inf\{\|(z-A)^{-1}\|^{-1} : z \in \partial\Omega\}$, then $\sigma(B) \cap \Omega \neq \varnothing$.*

Proof of Theorem 2. From the fact that $\delta_n \to 0$, we know that there is a sequence $\{A_n\}$ of norm-one hermitian matrices and a sequence $\{N_n\}$ of nilpotent matrices such that $\|A_n - N_n\| \to 0$. It follows that $\|N_n\| \to 1$, and we conclude that $\|A_n^2 - N_n^2\| \to 0$. Since $\|A_n^2\| = 1$ and N_n^2 is nilpotent, we can assume without loss of generality that $0 \le A_n \le 1$.

We remarked above that a norm limit of nilpotent operators must have connected spectrum. The A_n's, being finite matrices, cannot have connected spectrum; however, their spectra must try to fill out the interval $[0, 1]$ as $n \to \infty$.

Lemma 4. *For each t in* $[0, 1]$, $\operatorname{dist}(t, \sigma(A_n)) \le \|A_n - N_n\|$.

Proof. Suppose $0 < t < 1$, and assume that $t \notin \sigma(A_n)$. If we apply Proposition 3 with $A = A_n$, $B = N_n$, and Ω the disk centered at 1 with radius $1-t$, then the fact that $\sigma(N_n) \cap \Omega = \varnothing$ and the fact that $\inf\{\|(z-A)^{-1}\|^{-1} : z \in \partial\Omega\} = \operatorname{dist}(t, \sigma(A_n))$ implies the desired inequality. □

We next show how the operator M_x, multiplication by the independent variable x, on the space $L^2[0, 1]$ with Lebesgue measure is a norm limit of nilpotents. Later, we will modify the argument to handle the general case.

Lemma 5. *The operator M_x is a norm limit of nilpotent operators.*

Proof. Let $0 \le t_{n_1} \le t_{n_2} < \cdots < t_{n_{k(n)}} = 1$ be the distinct eigenvalues of A_n. Let $A_n^{(\infty)}$, $N_n^{(\infty)}$ denote, respectively, a direct sum of infinitely many copies of A_n, N_n. Define $f_n : [0, 1] \to [0, 1]$ to be the simple function taking the value t_{n_k} on the interval $[t_{n_{k-1}}, t_{n_k})$ (where $t_{n0} = 0$).

It follows from Lemma 4 that $\|x - f_n(x)\|_\infty \to 0$; whence $\|M_{f_n} - M_x\| \to 0$. However, $A_n^{(\infty)}$ and M_{f_n} are both diagonalizable operators with eigenvalues $t_{n_1}, t_{n_2}, \ldots, t_{n_{k(n)}}$, each having infinite multiplicity. Thus there is a unitary operator U_n such that $U_n^* A_n^{(\infty)} U_n = M_{f_n}$, for each $n \ge 1$. Hence

$$\begin{aligned}\|U_n^* N_n^{(\infty)} U_n - M_x\| &\le \|U_n^*[N_n^{(\infty)} - A_n^{(\infty)}]U_n\| \\ &\quad + \|U_n^* A_n^{(\infty)} U_n - M_{f_n}\| + \|M_{f_n} - M_x\| \\ &= \|N_n - A_n\| + \|M_{f_n} - M_x\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Since $U_n^* N_n^{(\infty)} U_n$ is a nilpotent operator for every n, we have proved that M_x is a norm limit of nilpotent operators. □

Lemma 6. *If T is a normal operator, $0 \in \sigma(T)$, and $\sigma(T)$ is connected, then there is a sequence $\{p_n\}$ of polynomials vanishing at 0 and and a sequence $\{W_n\}$ of unitary operators with $\|W_n^* p_n(M_x) W_n - T\| \to 0$.*

Proof. Suppose $\varepsilon > 0$, and let $\Omega = \{z + w : z \in \sigma(T), |w| < e\}$. Then Ω is open and connected; whence Ω is path-connected. Since $\sigma(T)$ is compact, we can find a finite subset $\mathscr{F} = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ of $\sigma(T)$ with $\lambda_1 = 0$ such that every point in $\sigma(T)$ is within ε of some point in $\mathscr{F}$. It follows from the spectral theorem that there is a diagonalizable operator D whose set of eigenvalues is $\mathscr{F}$, with each eigenvalue having infinite multiplicity, such that $\|T - D\| < \varepsilon$. Since Ω is path-connected, there is a continuous function $\psi : [0, 1] \to \Omega$ such that $\psi(0) = 0$ and the range of ψ includes $\mathscr{F}$. Since ψ is uniformly continuous on $[0, 1]$, there is a positive integer m such that $\|\psi - h\|_\infty < \varepsilon$ where h is the simple function taking on the value $\psi(k/m)$ on the interval $[k/m, (k+1)/m)$, for $0 \le k \le m - 1$. Hence

$$\|\psi(M_x) - h(M_x)\| = \|M_\psi - M_h\| < \varepsilon.$$

However, M_h is a diagonalizable operator, with each eigenvalue having infinite multiplicity, and such that every point in $\mathscr{F}$ is within ε of the set of eigenvalues M_h.

It follows that every eigenvalue of D is within ε of an eigenvalue of M_h, and that each eigenvalue of M_h is within 2ε of an eigenvalue of D. By rearranging eigenvalues, we can find a unitary operator W such that $\|W^* M_h W - D\| < 2\varepsilon$. Hence $\|W^* \psi(M_x) W - D\| < 3\varepsilon$. Since, by the Weierstrass approximation theorem, ψ is a uniform limit of polynomials on $[0, 1]$ (that vanish at 0 since ψ vanishes at 0), there is a polynomial p with $p(0) = 0$ such that $\|p - \psi\|_\infty < \varepsilon$. Hence,

$$\begin{aligned}\|W^* p(M_x) W - T\| &\le \|W^*[p(M_x) - \psi(M_x]W\| \\ &\quad + \|W^* \psi(M_x) W - D\| + \|D - T\| \\ &< \varepsilon + 3\varepsilon + \varepsilon = 5\varepsilon.\end{aligned}$$

This completes the proof of the lemma. □

The proof of Theorem 2 is completed by noting that, since the set $\mathscr{N}$ of nilpotent operators is closed under unitary equivalence and under evaluation of polynomials that vanish at 0, the same is true for the norm closure $\mathscr{N}^-$. Since $M_x \in \mathscr{N}^-$, it follows that, for every unitary operator W and for every polynomial p with $p(0) = 0$, we have $W^* p(M_x) W \in \mathscr{N}^-$. The preceding lemma clearly implies that $\mathscr{N}^-$ contains every normal operator whose spectrum is connected and contains 0. This completes the proof of Theorem 2.

It follows from the proof of Lemma 1 that we could take

$$N_n = [\pi \tau_n(T_{u,n}) / \log(n)]^2 \quad \text{and} \quad A_n = [\mathrm{Re}(N_n)]^2$$

in the proof of Theorem 2. This, modulo the problem of computing the eigenvectors and eigenvalues of A_n, gives a "concrete" construction of a sequence of nilpotents that converge in norm to M_x on $L^2[0, 1]$.

The description of the closure of the set of all nilpotent operators on a separable infinite-dimensional Hilbert space is a deep theorem of C. Apostol,

C. Foias, and D. Voiculescu [AFV]. A formula for the distance from an operator to the set of nilpotent operators is contained in [AFHV, 12.7].

ACKNOWLEDGMENT

The author wishes to thank the National Science Foundation for its support while this research was undertaken.

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