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LEVEL CROSSINGS OF A RANDOM POLYNOMIAL WITH HYPERBOLIC ELEMENTS

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ABSTRACT. This paper provides an asymptotic estimate for the expected number of K-level crossings of a random hyperbolic polynomial $g_1 \cosh x + g_2 \cosh 2x + \cdots + g_n \cosh nx$, where g_j $(j = 1, 2, \ldots, n)$ are independent normally distributed random variables with mean zero, variance one and K is any constant independent of x. It is shown that the result for $K = 0$ remains valid as long as $K \equiv K_n = O(\sqrt{n})$.

1. INTRODUCTION

Let (Ω, A, Pr) be a fixed probability space and let $\{g_j(\omega)\}_{j=1}^n$ be a sequence of independent identically distributed random variables defined on Ω. Although there has been considerable attention given to algebraic and trigonometric polynomials with coefficients g_j's, very little is known about the behaviour of the random hyperbolic polynomial,

$$P(x) \equiv P_n(x, \omega) = \sum_{j=1}^{n} g_j(\omega) \cosh jx. \tag{1.1}$$

Denote by $N_K(\alpha, \beta)$ the number of real roots of the equation $P(x) = K$ in the interval (α, β) and by $EN_K(\alpha, \beta)$ its expected value. The only literature that this author could find concerning EN is a report by Bharucha-Reid and Sambandham [1, p. 110] on an unpublished result of Das [4], where it is stated that for $K = 0$ and independent normally distributed coefficients with mean zero and variance one $EN_0(-\infty, \infty)$ is asymptotic to $(1/\pi)\log n$. This is interesting as it shows that EN_0 for random hyperbolic polynomials does not correspond with that of the random algebraic polynomial

$$F(x) \equiv F_n(x, \omega) = \sum_{j=1}^{n} g_j(\omega) x^j, \tag{1.2}$$

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nor with that of the random trigonometric polynomial

$$(1.3)\qquad T(x) = T_n(x, \omega) = \sum_{j=1}^{n} g_j(\omega) \cos jx .$$

From Kac [7] or Wilkins [10] we know that for the algebraic polynomial (1.2), $EN_0(-\infty, \infty) \sim (2/\pi)\log n$ is twice that of the hyperbolic case reported by Bharucha-Reid and Sambandham [1], while for the trigonometric case (1.3), $EN_0(0, 2\pi) \sim (2n/\sqrt{3})$ (see Das [3] and Wilkins [9]). Therefore it is of special interest to establish for the hyperbolic case which of the known patterns, if any, EN_K, for $K \neq 0$, will follow. One can expect that, because of the similarity of order of EN_0, the K-level crossing would be similar to that of the algebraic case. In Farahmand [6] it is shown that EN_K for the equation $F(x) = K$ is asymptotically reduced to $(1/\pi)\log(n/K^2)$ in the interval $(-1, 1)$ while its remains the same as $K = 0$ in the interval $(-\infty, 1) \cup (1, \infty)$ as long as $K \equiv K_n = O(\sqrt{n})$. For the trigonometric equation $T(x) = K$, however, Farahmand [5] shows $EN_K(0, 2\pi)$ remains asymptotic to $(2n/\sqrt{3})$. Our result here unexpectedly shows that the K-level crossing of the hyperbolic polynomial is similar to that of the trigonometric one. If one classifies the oscillation of different types of polynomials according to the behaviour of their real zeros, viz. the algebraic types with $EN_0 = O(\log n)$ and the trigonometric types with $EN_0 = O(n)$, it seems interesting to note that although random hyperbolic polynomials will fall into the first category, their properties of K-level crossings follow the second. We prove the following:

Theorem. *For any sequence of constants $K_n \equiv K$ such that $\{K^2/(n \log n)\}$ tends to zero as n tends to infinity the mathematical expectation of the number of real roots of the equation $P(x) = K$ satisfies*

$$EN_K(-\infty, \infty) \sim (1/\pi) \log n .$$

2. A formula for the expected number of crossings

Let

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-y^2/2)\, dt$$

and

$$\varphi(t) = d\Phi(t)/dt = (2\pi)^{-1/2} \exp(-t^2/2);$$

then by using the expected number of level crossings given by Cramér and Leadbetter [2, p. 285] for our equation $P(x) - K = 0$ we can obtain

$$(2.1)\qquad EN(\alpha, \beta) = \int_{\alpha}^{\beta} (B/A)(1 - C^2/A^2B^2)^{1/2} \varphi(-K/A) \cdot [2\varphi(\eta) + \eta\{2\Phi(\eta) - 1\}]\, dx ,$$

where

$$A^2 = \mathrm{var}\{P(x) - K\} = \sum_{j=1}^{n} \cosh^2 jx, \tag{2.2}$$

$$B^2 = \mathrm{var}\{P'(x)\} = \sum_{j=1}^{n} j^2 \sinh^2 jx, \tag{2.3}$$

$$C = \mathrm{cov}[\{P(x) - K\}, P'(x)] = \sum_{j=1}^{n} j \sinh jx \cosh jx, \tag{2.4}$$

and

$$\eta = -CK/A(A^2B^2 - C^2)^{1/2}.$$

Let $\Delta^2 = A^2B^2 - C^2$ and $\mathrm{erf}(x) = \int_0^x \exp(-t^2)\,dt$; then from (2.1) we can write the extension of a formula obtained by Kac [7] and Rice [8] for the case of $K = 0$ as

$$EN(\alpha, \beta) = I_1(\alpha, \beta) + I_2(\alpha, \beta), \tag{2.5}$$

where

$$I_1(\alpha, \beta) = \int_\alpha^\beta (\Delta/\pi A^2) \exp(-B^2K^2/2\Delta^2)\,dx \tag{2.6}$$

and

$$I_2(\alpha, \beta) = \int_\alpha^\beta (\sqrt{2}/\pi) KCA^{-3} \exp(-K^2/2A^2)\,\mathrm{erf}(KC/A\Delta\sqrt{2})\,dx. \tag{2.7}$$

We remark that although we are interested in $x \in (-\infty, \infty)$ it is sufficient to restrict our attention to the number of real roots for positive x only; since to each root of $P(x, \omega) = K$ in $(0, \infty)$ there corresponds a root of $P(x, \omega) = K$ in $(-\infty, 0)$, and conversely. Therefore $EN(-\infty, \infty) = 2EN(0, \infty)$. From (2.2)–(2.4) we obtain the following relations:

$$A^2 = (2n-1)/4 + \sinh(2n+1)x/4\sinh x, \tag{2.8}$$

$$\begin{aligned} B^2 = &-n(n+1)(2n+1)/12 + (2n+1)^2 \sinh(2n+1)x/16\sinh x \\ &- (2n+1)\cosh(2n+1)x \cosh x/8\sinh^2 x \\ &+ (2\cosh^2 x/\sinh^2 x - 1)\sinh(2n+1)x/16\sinh x, \end{aligned} \tag{2.9}$$

$$\begin{aligned} C = &(2n+1)\cosh(2n+1)x/8\sinh x \\ &- \sinh(2n+1)x \cosh x/8\sinh^2 x, \end{aligned} \tag{2.10}$$

and therefore

$$\begin{aligned} \Delta^2 = &\sinh^2(2n+1)x/64\sinh^4 x \\ &+ (2n-1)\sinh(2n+1)x(1+\cosh^2 x)/64\sinh^3 x \\ &- (4n^2-1)\cosh(2n+1)x\cosh x/32\sinh^2 x \\ &+ (2n+1)(8n^2-4n-3)\sinh(2n+1)x/192\sinh x \\ &- (2n+1)^2/64\sinh^2 x - n(n+1)(4n^2-1)/48. \end{aligned} \tag{2.11}$$

3. Proof of the theorem

First we let x be the interval $((\log n)^{1/2}/n, 1)$. As it turns out this interval will make the main contribution to the number of real roots. To find the dominant terms in (2.2)–(2.4) we observe that in this interval

$$\coth x < e/x < en(\log n)^{-1/2},$$

and therefore the derivative of $f_{n,p}(x) = (\sinh nx)(\sinh x)^{-p}$ is positive for $p = 1, 2, 3$, and for n sufficiently large. Hence

$$\begin{aligned} f_{n,p}(x) &\geq \sinh\{(\log n)^{1/2}\}[\sinh\{(\log n)^{1/2}/n\}]^{-p} \\ &\geq \sinh\{(\log n)^{1/2}\}[(n/4)(\log n)^{-1/2}]^p. \end{aligned} \tag{3.1}$$

Use has been made of the fact that $\sinh x \leq 4x$ in $(0, 1)$. Since $(\log n)^{1/2}/n$ is a decreasing function of n, (3.1) will remain valid for n replaced by $2n+1$. Hence for all $n \geq 49$ and $p = 1, 2, 3$ from (3.1) we can obtain

$$f_{2n+1,p}(x) \geq (n^p/3.254)(\log n)^{-p/2}\exp\{2(\log n)^{1/2}\}. \tag{3.2}$$

Now from (2.8)–(2.11) and (3.2), for all sufficiently large n, and since $\sinh x \geq x/4$ in $(0, 1)$ we can show

$$A^2 = [\{\sinh(2n+1)x\}/4\sinh x]\{1 + O(\log n)^{-1}\}, \tag{3.3}$$

$$B^2 = [(2n+1)^2\{\sinh(2n+1)x\}/16\sinh x]\{1 + O(\log n)^{-1/2}\}, \tag{3.4}$$

$$C = [(2n+1)\{\cosh(2n+1)x\}/8\sinh x]\{1 + O(\log n)^{-1/2}\}, \tag{3.5}$$

and

$$A^2B^2 - C^2 = [\{\sinh(2n+1)x\}/8\sinh^2 x]^2\{1 + O(\log n)^{-1}\}. \tag{3.6}$$

In the following for n sufficiently large we evaluate $I_1((\log n)^{1/2}/n, 1)$. To this end from (2.6) and (3.3)–(3.6) we have

$$\begin{aligned} &I_1((\log n)^{1/2}/n, 1) \\ &\quad = (2\pi)^{-1}[1 + O(\log n)^{-1}] \\ &\qquad \cdot \int_{(\log n)^{1/2}/n}^{1} (\operatorname{csch} x)\exp\{-2K^2(2n+1)^2\sinh^3 x/ \\ &\qquad\qquad \sinh(2n+1)x[1 + O(\log n)^{-1/2}]\}\,dx \\ &\quad = (2\pi)^{-1}[1 + O(\log n)^{-1}]\int_{(\log n)^{1/2}/n}^{1}(\operatorname{csch} x)\,dx \\ &\qquad + O\left[K^2(2n+1)^2\int_{(\log n)^{1/2}/n}^{1}\{\sinh^2 x/\sinh(2n+1)x\}\,dx\right]. \end{aligned} \tag{3.7}$$

The first term appearing on the right-hand side of (3.7) can be evaluated as

$$\begin{aligned} &(2\pi)^{-1}[1 + O(\log n)^{-1}]\{\log(\tanh 1/2) - \log[\tanh\{(\log n)^{1/2}/2n\}]\} \\ &\quad = (2\pi)^{-1}[1 + O(\log n)^{-1}]\{O(1) + \log\{n(\log n)^{-1/2}\} \\ &\qquad\qquad - \log[1 + O(n^{-2}\log n)]\} \\ &\quad = (2\pi)^{-1}\{\log n - (1/2)\log(\log n) + O(1)\}. \end{aligned} \tag{3.8}$$

Now we show that the second term appearing on the right-hand side of (3.7) is small compared with the value obtained in (3.8). To this end we write this term as

$$\begin{aligned}
&O\left[K^2(2n+1)^2\int_{(\log n)^{1/2}/(n+1/2)}^{1}\{x^2\operatorname{csch}(2n+1)x\}\,dx\right]\\
(3.9)\qquad &= O\left[K^2(2n+1)^{-1}\int_{2(\log n)^{1/2}}^{2n+1}u^2\operatorname{csch}u\,du\right]\\
&= O[K^2(2n+1)^{-1}].
\end{aligned}$$

Therefore, since $K = o(n\log n)^{1/2}$, from (3.7)–(3.9) we have $(2\pi)^{-1}\log n$ as the asymptotic value for $I_1((\log n)^{1/2}/n, 1)$. Now we show $I_1(0, (\log n)^{1/2}/n)$ is small compared with this asymptotic value. For this range of x the dominant term for A^2, B^2 and C cannot be found. However, since from (2.8)

$$A^2 > \{\sinh(2n+1)x\}/4\sinh x$$

and since $u\coth u$ is an increasing function of u we can have

$$\begin{aligned}
&(B^2/A^2) \le n(n+1) + (1/2)\coth x\{\coth x - (2n+1)\coth(2n+1)x\}\\
(3.10)\qquad &\qquad - \{n(n+1)(2n+1)\sinh x\}/\{3\sinh(2n+1)x\}\\
&\qquad < n(n+1) < (n+1/2)^2.
\end{aligned}$$

Therefore for $x \ge 0$ and for all $n \ge 2$ from (3.10) we can obtain

$$(3.11)\qquad (\Delta/A^2) \le (B/A) < n + 1/2.$$

Therefore from (2.6), (3.11) and for all sufficiently large n we can obtain

$$(3.12)\qquad I_1(0, (\log n)^{1/2}/n) = O((\log n)^{1/2}).$$

To estimate an upper limit for I_2 we note that since $C = (1/2)d(A^2)/dx$ from (2.7) we can write

$$\begin{aligned}
&I_2(0,\infty) < (2\pi)^{-1/2}\int_0^\infty |KC|A^{-3}\,dx\\
(3.13)\qquad &\qquad = |K|(2\pi)^{-1/2}\int_{n^{1/2}}^\infty A^{-2}\,dA\\
&\qquad = O(K/n^{1/2}).
\end{aligned}$$

Now it only remains to consider the case of $x \ge 1$ for I_1. From (2.8) and (2.11) and for sufficiently large n we have

$$\Delta^2 < \sinh^2(2n+1)x/16\sinh^4 x$$

and

$$A^2 > \sinh(2n+1)x/4\sinh x.$$

Therefore for all positive x,

$$(3.14)\qquad (\Delta/A^2) < \operatorname{csch} x.$$

Hence from (2.6) and (3.14) we obtain

$$\begin{aligned}
&I_1(1,\infty) < (\pi)^{-1}\int_1^\infty (\Delta/A^2)\,dx\\
(3.15)\qquad &\qquad < (\pi)^{-1}\int_1^\infty \operatorname{csch} x\,dx = (\pi)^1\log\{\coth(1/2)\}.
\end{aligned}$$

Finally from (3.8), (3.9), (3.12), (3.13), and (3.15) we have proof of the theorem.

4. Remark

By looking at the proof it is apparent that although in the interval of $(-1, 1)$ the hyperbolic polynomial has asymptotically as many roots as the algebraic polynomial, outside this interval, unlike the algebraic case, the hyperbolic polynomial does not possess any sizeable roots. Perhaps this is caused by (exponentially) fast increases (decreases) of the terms in $(-\infty, -1) \cup (1, \infty)$ which makes the cancellation in this type of polynomial difficult.

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