UNIQUE INTEGRABILITY OF CONTINUOUS k-PLANE FIELDS

PATRICK D. McSWIGGEN

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ABSTRACT. It is shown that unique integrability of a continuous k-plane field is a generic property for plane field satisfying a "topological bracket condition".

This note was motivated by the general observation that in a dynamical system it is quite common for there to occur an invariant plane field which is only continuous, but which nevertheless integrates uniquely. (For example, the stable and unstable planes of an Anosov diffeomorphism). For any given example there are, of course, more immediate explanations of why the integral manifolds should be unique. However, it seems surprising considered just in terms of the general problem of integrating continuous plane fields. In this note we show that a result on the generic integrability of continuous vector fields in [4] (see [2, p. 121]) can be extended to plane fields, and this shows that, in general, unique integrability of continuous plane fields is not uncommon.

Let M be a not necessarily compact manifold of dimension n. (M could, for example, be an open ball.) Let $\mathcal{X}_k^r(M)$, $0 < k < n$, be the space of C^r k-plane fields on M, where M is assumed to be a manifold that admits such plane fields. That is, $\mathcal{X}_k^r(M)$ is the space of C^r sections of the Grassmann bundle, $G_k(M)$, of k-planes tangent to M. As a function space we assume $\mathcal{X}_k^r(M)$ has been endowed with either the weak or the strong topology [3]. For $r = 0$, the weak topology on $\mathcal{X}_k^0(M)$ is the topology of uniform convergence on compact sets.

If $E \in \mathcal{X}_k^r(M)$, then E is a C^r subbundle of TM of dimension k. E is *integrable* if for every $x \in M$ there is a (local) C^{r+1} submanifold $N \subset M$ through x whose tangent bundle is E restricted to N. We say that a C^1 plane field E is *involutive* if for any two C^1 vector fields X and Y taking values in E, $[X, Y]$ again takes values in E. For C^0 plane fields we make the following definition.

Definition. We say $E \in \mathcal{X}_k^0(M)$ is *topologically involutive* if E is the weak limit of smooth (at least C^1) involutive plane fields.

Frobenius' Theorem [1] states that for $r \geq 1$, E is integrable if and only if it is involutive, in which case it is uniquely integrable. For continuous plane fields,

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not all integrable plane fields need to be topologically involutive. However, we do have the following.

Proposition. *If $E \in \mathscr{X}_k^0(M)$ is topologically involutive, then E is integrable.*

Proof. Let $x \in M$. If E is topologically involutive, then there exist smooth involutive plane fields $E_i \to E$ uniformly near x. Let N_i be the (local) integral manifold of E_i through x, and let V_x be a plane in $T_x M$ complementary to E_x. Since E is continuous, there exists a neighborhood U of zero in E_x such that for each i (in exponential coordinates), N_i over U is the graph of a C^1 function $g_i: U \to V_x$. The family $\{g_i\}$ is equicontinuous and bounded, and hence precompact. Therefore, there exists a subsequence converging to some C^1 function $g: U \to V_x$. The graph of g is then an integral manifold of E through x. $\square$

By definition, the subset of $\mathscr{X}_k^0(M)$ of topologically involutive plane fields is weakly closed. A weakly closed subset in either the weak or strong topology is a Baire space [3]. It is well known that a continuous plane field which is integrable need not be uniquely integrable. However, the following shows that unique integrability is a generic property of topologically involutive plane fields.

Theorem. *The set of uniquely integrable continuous k-plane fields on M is residual in the topologically involutive k-plane fields.*

Proof. Let $\{K_i\}$ be a countable cover of M by compact sets, and let $\mathscr{I}(K_i)$ be the set of topologically involutive plane fields on M with the following property. For any $E \in \mathscr{I}(K_i)$, if N_1 and N_2 are any local integral manifolds of E through a point $x \in K_i$, then N_1 and N_2 agree on a neighborhood of x. Clearly, $\bigcap_i \mathscr{I}(K_i)$ is the set of uniquely integrable plane fields. Therefore, it will be sufficient to show that $\mathscr{I}(K)$ is residual for any compact $K \subset M$.

A set is residual if and only if it is residual relative to each element of some open cover. That is, its intersection with each element is residual in that element. Therefore, it will be sufficient to show $\mathscr{I}(K)$ is residual relative to some neighborhood of any fixed, arbitrary topologically involutive plane field E.

Fix a topologically involutive plane field E. For $x \in K$, choose local coordinates on a neighborhood U of x identifying x with the origin and E_x with $\mathbb{R}^k \times \{0\}$. By identifying an element $\mathbf{A} \in L(\mathbb{R}^k, \mathbb{R}^{n-k})$ with the plane graph $(\mathbf{A}) := \{(v, \mathbf{A}v)\} \in G_k(M)$ in these local coordinates, we can define a neighborhood W of E_x in $G_k(M)$ by $W = U \times \{\mathbf{A}: \|\mathbf{A}\| < 1\}$. E is continuous, so if U is sufficiently small, $E(U) \subset W$. Let D_r^m denote the closed disk of radius r in $\mathbb{R}^m$, and choose $d > 0$ such that $B := D_{2d}^k \times D_{2d}^{n-k} \subset U$. Let $C := D_d^k \times D_d^{n-k}$.

Choose a finite collection $\{(U_i, W_i, B_i, C_i)\}$ as above such that $\bigcup C_i \supset K$, and define a neighborhood $\mathscr{N}$ of E by $\mathscr{N} = \{F: \forall i \; F(B_i) \subset W_i\}$. We want to show $\mathscr{I}(K)$ is residual relative to $\mathscr{N}$. Let $F \in \mathscr{N}$, $x \in C_i$ for some i, and let N_1 and N_2 be any two local integral manifolds of F through x. Write x in the local coordinates on U_i as $(u_0, v_0) \in D_d^k \times D_d^{n-k}$. By our choice of $\mathscr{N}$, $N_1 \cap \operatorname{int} B_i$ is the graph of a C^1 function $g_1: V_1 \to \mathbb{R}^{n-k}$, where $V_1 \subset D_{2d}^k$ is the projection (in local coordinates) of N_1 into $\mathbb{R}^k$, and is some neighborhood of u_0. Likewise, N_2 is the graph of some $g_2: V_2 \to \mathbb{R}^{n-k}$. Set $\hat{r} = \sup\{r \le d: \text{disk centered on } u_0 \text{ with radius } r \subset V_1 \cap V_2\}$. Let

$\delta_i[N_1, N_2] := \sup\{\|(g_1 - g_2)(u)\|/\|u - u_0\| : \|u - u_0\| \leq 1/2\hat{r}\}$. This is finite since $(g_1 - g_2)$ is C^1, and N_1, N_2 both contain $x \Rightarrow g_1(u_0) = v_0 = g_2(u_0)$. $\delta_i[N_1, N_2]$ measures the rate at which N_1 and N_2 diverge (as measured by their rate of vertical separation in local coordinates, but only over a disk which is common to both). Note that $\delta_i[N_1, N_2]$ is well defined for any two integral manifold of F through $x \in C_i$.

For $\rho > 0$ define a set $\mathscr{S}_\rho \subset \mathscr{N}$ as follows. $F \in \mathscr{S}_\rho$ iff for any i, any $x \in C_i$, and any integral manifolds N_1, N_2 of F through x, $\delta_i[N_1, N_2] < \rho$. Suppose $F \in \bigcap_{n \geq 1} \mathscr{S}_{1/n}$, and let N_1 and N_2 be a fixed pair of integral manifolds of F through $x \in K$. Then $x \in C_i$ for some i, and so $\delta_i[N_1, N_2] < 1/n$ for all n. Hence, $g_1(u) = g_2(u)$ for all u with $\|u - u_0\| \leq 1/2\hat{r}$, and $\hat{r}$ depends only on the overlap $V_1 \cap V_2$. Therefore, N_1 and N_2 agree on a neighborhood of x. Since this holds for any $x \in K$, we have $\bigcap_{n \geq 1} \mathscr{S}_{1/n} \subset \mathscr{S}(K)$. We claim that for any $\rho > 0$, $\mathscr{S}_\rho$ contains an open and dense set, which will complete the proof.

By the definition of topologically involutive plane fields, the C^1 involutive plane fields are dense in $\mathscr{N}$. Let $\widehat{F} \in \mathscr{N}$ be C^1. Since $\widehat{F}$ is uniquely integrable, $\widehat{F} \in \mathscr{S}_\rho$. For $\varepsilon > 0$, let $\mathscr{O}_\varepsilon(\widehat{F})$ be the neighborhood of $\widehat{F}$ of those $F \in \mathscr{N}$ which, on each B_i, differ from $\widehat{F}$ in local coordinates by at most ε. That is, for any U_i, if $\hat{\mathbf{A}}$, $\mathbf{A} \in C^0(U_i, L(\mathbb{R}^k, \mathbb{R}^{n-k}))$ are defined in local coordinates by $\widehat{F}_y = \text{graph}(\hat{\mathbf{A}}_y)$ and $F_y = \text{graph}(\mathbf{A}_y)$, then $\|\hat{\mathbf{A}}_y - \mathbf{A}_y\| < \varepsilon$ for all $y \in B_i$. We want to show $\mathscr{O}_\varepsilon(\widehat{F}) \subset \mathscr{S}_\rho$ for ε sufficiently small.

Suppose $F \in \mathscr{O}_\varepsilon(\widehat{F})$ and $x \in C_i$. Let N_1 and N_2 be two integrable manifolds of F through x. We want $\delta_i[N_1, N_2] < \rho$ for ε small. Let $\hat{\mathbf{A}}$ and $\mathbf{A}$ be as above. Since $\hat{\mathbf{A}}$ is C^1 (and integrable), there is a C^1 foliation on $D_{2d}^k \times D_{2d}^{n-k}$ integrating $\hat{\mathbf{A}}$ whose leaves are all graphs over D_{2d}^k. Therefore, there is a C^1 foliation chart $H: D_{2d}^k \times D_{2d}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ taking these leaves to flat planes.

Let (u_0, v_0) be the coordinates of x. Let N_1 and N_2 be the graphs of g_1 and g_2. For $j = 1, 2$, we have $g_j(u_0) = v_0$, and $(Dg_j)_u = \mathbf{A}_{(u, g_ju)}$ for all $u \in V_j$. Let $\hat{r}$ be defined as above. Fix u with $\|u - u_0\| \leq 1/2\hat{r}$, and for $t \in [0, 1]$, define $\mu(t) := u_0 + t(u - u_0)$. Let $\gamma_j(t) := (\mu(t), g_j(\mu(t)))$, $j = 1, 2$. $\gamma_j(t)$ is a curve in the graph of g_j from (u_0, v_0) to $(u, g_j(u))$. Since H is a foliation chart, $(DH)_{(u,v)}$ takes the tangent plane at $(u, v) = \text{graph}(\hat{\mathbf{A}}_{(u,v)})$ to $\mathbb{R}^k \times \{0\}$. Consequently, $(DH)_{(u,v)}[\mathbf{I} \ \hat{\mathbf{A}}_{(u,v)}]^\top = [\mathbf{I} \ \mathbf{0}]^\top$ for all (u, v), and in particular,

$$(DH)_{\gamma_1(t)}[\mathbf{I} \ \hat{\mathbf{A}}_{\gamma_1(t)}]^\top = (DH)_{\gamma_2(t)}[\mathbf{I} \ \hat{\mathbf{A}}_{\gamma_2(t)}]^\top.$$

Therefore, we have the following:

$$\|(g_2 - g_1)(u)\| = \|(u, g_2 u) - (u, g_1 u)\|$$
$$\leq \text{Lip}(H^{-1})\|H(u, g_2 u) - H(u, g_1 u)\|,$$

$$\|H(u, g_2 u) - H(u, g_1 u)\|$$
$$\leq \int_0^1 \|(DH)_{\gamma_2(t)}[\mathbf{I} \ \mathbf{A}_{\gamma_2(t)}]^\top - (DH)_{\gamma_1(t)}[\mathbf{I} \ \mathbf{A}_{\gamma_1(t)}]^\top\| \ \|u - u_0\| \, dt,$$

and

$$\|(DH)_{\gamma_2(t)}[\mathbf{I}\ \mathbf{A}_{\gamma_2(t)}]^\top - (DH)_{\gamma_1(t)}[\mathbf{I}\ \mathbf{A}_{\gamma_1(t)}]^\top\|$$

$$\leq \|(DH)_{\gamma_2(t)}[\mathbf{I}\ \mathbf{A}_{\gamma_2(t)}]^\top - (DH)_{\gamma_2(t)}[\mathbf{I}\ \hat{\mathbf{A}}_{\gamma_2(t)}]^\top\|$$

$$+ \|(DH)_{\gamma_1(t)}[\mathbf{I}\ \hat{\mathbf{A}}_{\gamma_1(t)}]^\top - (DH)_{\gamma_1(t)}[\mathbf{I}\ \mathbf{A}_{\gamma_1(t)}]^\top\|$$

$$\leq \|(DH)_{\gamma_2(t)}\|\,\|\mathbf{A}_{\gamma_2(t)} - \hat{\mathbf{A}}_{\gamma_2(t)}\| + \|(DH)_{\gamma_1(t)}\|\,\|\mathbf{A}_{\gamma_1(t)} - \hat{\mathbf{A}}_{\gamma_1(t)}\|$$

$$\leq 2\,\mathrm{Lip}(H)\sup\{\|\mathbf{A}_{(u,v)} - \hat{\mathbf{A}}_{(u,v)}\| : \|u\|,\ \|v\| \leq 2d\} < 2\varepsilon\,\mathrm{Lip}(H).$$

Consequently, $\|(g_2 - g_1)(u)\| < 2\varepsilon\,\mathrm{Lip}(H)\,\mathrm{Lip}(H^{-1})\|u - u_0\| \leq \rho\|u - u_0\|$ for ε sufficiently small. Since H depended only on $\widehat{F}$ and the fixed local coordinates on U_i, we can choose one ε so that the above inequalities hold in each U_i. Hence, $\mathscr{O}_\varepsilon(\widehat{F}) \subset \mathscr{S}_\rho$.

Since the set of $C^1 F$ is dense in $\mathscr{S}_\rho$, this implies $\mathscr{S}_\rho$ contains an open and dense set. Consequently, $\mathscr{S}(K)$ is residual, which completes the proof. $\square$

References

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DEPARTMENT OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221-0025
E-mail address: `pat.mcswiggen@uc.edu`