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STOLARSKY'S INEQUALITY WITH GENERAL WEIGHTS

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ABSTRACT. Recently Stolarsky proved that the inquality

$$(*) \qquad \int_0^1 g(x^{1/(a+b)})\,dx \geq \int_0^1 g(x^{1/a})\,dx \int_0^1 g(x^{1/b})\,dx$$

holds for every $a, b > 0$ and every nonincreasing function on $[0, 1]$ satisfying $0 \leq g(u) \leq 1$. In this paper we prove a weighted version of this inequality. Our proof is based on a generalized Chebyshev inequality. In particular, our result shows that the inequality $(*)$ holds for every function g of bounded variation. We also generalize another inequality by Stolarsky concerning the Γ-function.

The following remarkable inequality, recently proved by Stolarsky, has interesting applications (see [6]).

Proposition A. *If $0 \leq g(x) \leq 1$ and g is nonincreasing on $[0, 1]$, then, for all positive numbers a and b, it holds that*

$$(1) \qquad \int_0^1 g(x^{1/(a+b)})\,dx \geq \int_0^1 g(x^{1/a})\,dx \int_0^1 g(x^{1/b})\,dx.$$

Stolarsky also observed that by introducing the quotient $Q(g, p)$ as

$$Q(g, p) = \int_0^1 g(x)x^{p-1}\,dx \Big/ \int_0^1 x^{p-1}\,dx,$$

we can make a change of variables and formulate (1) as

$$Q(g, a+b) \geq Q(g, a)Q(g, b).$$

In this paper we will generalize this result by considering the quotient

$$Q(g, w) = \int_0^1 g(x)w(x)\,dx \Big/ \int_0^1 w(x)\,dx,$$

where w is a nonnegative and integrable weight function on $[0, 1]$, and by permitting that g is an arbitrary function of bounded variation on $[0, 1]$ (see Theorem 1). Our proof is different from that in [6] and based upon some variants of Chebyshev's inequality.

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We consider nonnegative and integrable weight functions w_1, w_2, and w_3 on $[0, 1]$ and introduce the notation

$$W_i(x) = \int_0^x w_i(t)\,dt \Big/ \int_0^1 w_i(t)\,dt, \qquad i = 1, 2, 3.$$

Our main result reads:

Theorem 1. *If g is a function of bounded variation on $[0, 1]$ such that $0 \le g(1) \le g(x) \le g(0)$ for all $x \in (0, 1)$ and if*

$$W_1(x)W_2(x) = W_3(x) \quad \textit{for all } x \in [0, 1], \tag{2}$$

then

$$g(0)Q(g, w_3) \ge Q(g, w_1)Q(g, w_2). \tag{3}$$

Proof. Inequality (3) reduces to an equality for every $g \equiv C$, and thus we may, without loss of generality, assume that $g(1) < g(0)$. First we make a partial integration and find that

$$\begin{aligned} Q(g, w_i) &= \int_0^1 g(x)w_i(x)\,dx \Big/ \int_0^1 w_i(x)\,dx \\ &= \left\{ \left[g(x) \int_0^x w_i(t)\,dt \right]_0^1 - \int_0^1 \left(\int_0^x w_i(t)\,dt \right) dg(x)\,dx \right\} \Big/ \int_0^1 w_i(x)\,dx, \end{aligned}$$

i.e.,

$$Q(g, w_i) = g(1) - \int_0^1 W_i(x)\,dg(x), \qquad i = 1, 2, 3. \tag{4}$$

Next we note that

$$\begin{aligned} \int_0^1 W_i(x)\,dg(x) - g(1) + g(0) &= \int_0^1 (W_i(x) - 1)\,dg(x) \\ &= [(W_i(x) - 1)g(x)]_0^1 - \int_0^1 g(x)\,dW_i(x) \\ &= g(0) - \int_0^1 g(x)\,dW_i(x) = \int_0^1 [g(0) - g(x)]\,dW_i(x) \ge 0 \end{aligned}$$

and

$$\begin{aligned} \int_0^1 W_i(x)\,dg(x) &= [W_i(x)g(x)]_0^1 - \int_0^1 g(x)\,dW_i(x) = g(1) - \int_0^1 g(x)\,dW_i(x) \\ &= \int_0^1 [g(1) - g(x)]\,dW_i(x) \le 0. \end{aligned}$$

By combining these estimates we get

$$0 \le \frac{1}{g(1) - g(0)} \int_0^1 W_i(x)\,dg(x) \le 1, \qquad i = 1, 2, 3. \tag{5}$$

Now we recall that the discrete Chebyshev inequality says that

$$(p_1+p_2)(p_1a_1b_1+p_2a_2b_2) \geq (p_1a_1+p_2a_2)(p_1b_1+p_2b_2), \tag{6}$$

whenever $a_1 \geq a_2$, $b_1 \geq b_2$, and $p_1, p_2 \geq 0$ (see [3, p. 43]). In fact, inequality (6) follows from the following equalities:

$$\begin{aligned}(a_1-a_2)(b_1-b_2)p_1p_2 &= a_1b_1p_1^2+(a_1b_1+a_2b_2)p_1p_2+a_2b_2p_2^2\\ &\quad - a_1b_1p_1^2-(a_1b_2+a_2b_1)p_1p_2-a_2b_2p_2^2\\ &= (p_1+p_2)(a_1b_1p_1+a_2b_2p_2)\\ &\quad -(a_1p_1+a_2p_2)(b_1p_1+b_2p_2).\end{aligned}$$

In view of (5) we can apply (6) with $p_1 = g(1)$, $p_2 = g(0)-g(1)$, $a_1 = b_1 = 1$,

$$a_2 = \frac{1}{g(1)-g(0)}\int_0^1 W_1(x)\,dg(x), \quad \text{and} \quad b_2 = \frac{1}{g(1)-g(0)}\int_0^1 W_2(x)\,dg(x)$$

to find that

$$\begin{aligned}&g(0)\left[g(1)-\frac{1}{g(1)-g(0)}\int_0^1 W_1(x)\,dg(x)\int_0^1 W_2(x)\,dg(x)\right]\\ &\geq \left[g(1)-\int_0^1 W_1(x)\,dg(x)\right]\left[g(1)-\int_0^1 W_2(x)\,dg(x)\right]\\ &= Q(g, w_1)Q(g, w_2).\end{aligned} \tag{7}$$

We also need the following inequality of Chebyshev type of independent interest:

$$\int_0^1 W_1(x)\,dg(x)\int_0^1 W_2(x)\,dg(x) \leq \int_0^1 dg(x)\int_0^1 W_1(x)W_2(x)\,dg(x). \tag{8}$$

Inequality (8) is the classical Chebyshev inequality for integrals with the positive measure $d(-g)$ provided that g is a nonincreasing function (see [4, p. 40]). In our case g has only finite variation and we must use a generalized form of the Chebyshev inequality. In fact, now (8) is a special case of a result by Fink and Jodeit (see [1, Theorem 2] and [2, Theorem 2]). Here we present another independent proof of (8) for functions of bounded variation.

For $x \in [0, 1]$ we let

$$V(x) = \int_0^x W_2(t)\,dg(t)\int_0^1 dg(t) - \int_0^1 W_2(t)\,dg(t)\int_0^x dg(t).$$

Then $V(1) = 0$ and

$$\begin{aligned}
V(x) &= [g(1) - g(0)] \int_0^x W_2(t)\, dg(t) - [g(x) - g(0)] \int_0^1 W_2(t)\, dg(t) \\
&= [g(1) - g(x)] \int_0^x W_2(t)\, dg(t) - [g(x) - g(0)] \int_x^1 W_2(t)\, dg(t) \\
&= [g(1) - g(x)] \left\{ [W_2(t) g(t)]_0^x - \int_0^x g(t)\, dW_2(t) \right\} \\
&\quad - [g(x) - g(0)] \left\{ [W_2(t) g(t)]_x^1 - \int_x^1 g(t)\, dW_2(t) \right\} \\
&= [g(1) - g(x)] \left[W_2(x) g(x) - \int_0^x g(t)\, dW_2(t) \right] \\
&\quad - [g(x) - g(0)] \left[g(1) - W_2(x) g(x) - \int_x^1 g(t)\, dW_2(t) \right] \\
&= [g(1) - g(x)] \left[W_2(x) g(0) - \int_0^x g(t)\, dW_2(t) \right] \\
&\quad - [g(x) - g(0)] \left[g(1) - g(1) W_2(x) - \int_x^1 g(t)\, dW_2(t) \right] \\
&= [g(1) - g(x)] \int_0^x [g(0) - g(t)]\, dW_2(t) \\
&\quad - [g(x) - g(0)] \int_x^1 [g(1) - g(t)]\, dW_2(t) \\
&\le 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^1 dg(x) \int_0^1 W_1(x) W_2(x)\, dg(x) - \int_0^1 W_1(x)\, dg(x) \int_0^1 W_2(x)\, dg(x) \\
&\quad = \int_0^1 W_1(x)\, dV(x) \\
&\quad = [W_1(x) V(x)]_0^1 - \int_0^1 V(x)\, dW_1(x) = V(1) - \int_0^1 V(x)\, dW_1(x) \ge 0,
\end{aligned}$$

and the inequality (8) is proved.

Finally, by combining (7) with (8) and using (2) with (4) we find that

$$\begin{aligned}
&Q(g, w_1) Q(g, w_2) \\
&\quad \le g(0) \left[g(1) - \frac{1}{g(1) - g(0)} \int_0^1 W_1(x)\, dg(x) \int_0^1 W_2(x)\, dg(x) \right] \\
&\quad \le g(0) \left[g(1) - \int_0^1 W_1(x) W_2(x)\, dg(x) \right] \\
&\quad = g(0) \left[g(1) - \int_0^1 W_3(x)\, dg(x) \right] = g(0) Q(g, w_3),
\end{aligned}$$

and the proof is complete.

Remark 1. Inequality (3) is sharp, and it reduces to an equality if $g(x) = C\chi_{[0,a]}(x)$ for any $a \in [0, 1]$.

Remark 2. Our proof above shows that if we restrict ourselves to nonincreasing functions g, then (3) holds even if assumption (2) is replaced by the condition

$$W_1(x)W_2(x) \leq W_3(x).$$

Remark 3. In the proof of (8) we found the following identity:

$$\begin{aligned}\int_0^1 dg(x) \int_0^1 W_1(x)W_2(x)\,dg(x) - \int_0^1 W_1(x)\,dg(x) \int_0^1 W_2(x)\,dg(x)\\ = \int_0^1 \left([g(x) - g(1)] \int_0^x [g(0) - g(t)]\,dW_2(t)\right) dW_1(x)\\ + \int_0^1 \left([g(0) - g(x)] \int_x^1 [g(t) - g(1)]\,dW_2(t)\right) dW_1(x).\end{aligned}$$

This identity was discussed also in [7], but our proof is different (cf. also [5]). Also, Fink and Jodeit obtained their result by first proving a suitable identity.

Remark 4. Obviously Theorem 1 can be generalized in various ways. Here we mention the following possibility: Let g be as in Theorem 1, let the weights $w_1, w_2, \ldots, w_n$ be positive and integrable functions on $[0, 1]$, and let

$$W_i(x) = \int_0^x w_i(t)\,dt, \quad i = 1, 2, \ldots, n, \qquad \text{and} \qquad W_{n+1}(x) = \prod_1^n W_i(x).$$

Then, by using Theorem 1 and induction, we obtain the following generalization of (3):

$$(g(0))^{n-1} Q(g, w_{n+1}) \geq \prod_1^n Q(g, w_i).$$

Remark 5. Let g be a nonincreasing and nonnegative function on $[0, 1]$. Then a simple calculation shows that $Q(g, p)$ is a log-convex function. Now, by using well-known inequalities for log-convex or convex functions, we obtain various inequalities for the Stolarsky quotient $Q(g, a)$, e.g., the following: If $0 < a < b < c$, then

$$(Q(g, b))^{c-a} \leq (Q(g, a))^{c-b} (Q(g, c))^{b-a}.$$

Remark 6. Stolarsky presented some interesting applications of his inequality (1). In particular he pointed out a new inequality for the Γ-function. We remark that similarly to the proof of Theorem 1 we can prove this inequality directly by using the Chebyshev inequality. We finish this paper by proving the following more general result, which Stolarsky [6] proved only for $n = 2$.

Theorem 2. *Let a_i, $i = 1, 2, \ldots, n$, and x be positive numbers. Then*

$$G(x) = \frac{\Gamma(x + \sum_1^n a_i)\Gamma(x)^{n-1}}{\prod_1^n \Gamma(x + a_i)}$$

is a nonincreasing function of x.

Proof. Our proof is based upon the Chebyshev inequality applied $(n-1)$ times to the increasing functions $f_i(t) = t^{a_i}$, $i = 1, 2, \dots, n$ and with the weight $w(t) = t^{x-1}(1-t)^{y-1}$. In this way we find that

$$\left(\int_0^1 t^{x-1}(1-t)^{y-1}\,dt\right)^{n-1} \int_0^1 \prod_1^n t^{a_i} t^{x-1}(1-t)^{y-1}\,dt$$
$$\geq \prod_1^n \int_0^1 t^{a_i} t^{x-1}(1-t)^{y-1}\,dt,$$

i.e., $B(x, y)^{n-1} B\left(x + \sum_1^n a_i, y\right) \geq \prod_1^n B(x + a_i, y)$. Therefore, by using the fact that $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, $x, y > 0$,

$$\frac{\Gamma(x)^{n-1}\Gamma(x + \sum_1^n a_i)}{\Gamma(x+y)^{n-1}\Gamma(x + y + \sum_1^n a_i)} \geq \prod_1^n \frac{\Gamma(x + a_i)}{\Gamma(x + y + a_i)},$$

and this inequality can be rewritten as

$$\frac{\Gamma(x)^{n-1}\Gamma(x + \sum_1^n a_i)}{\prod_1^n \Gamma(x + a_i)} \geq \frac{\Gamma(z)^{n-1}\Gamma(z + \sum_1^n a_i)}{\prod_1^n \Gamma(z + a_i)},$$

where $x \leq z = x + y$, and the proof is complete.

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