A PALEY-WIENER THEOREM FOR FRAMES

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Abstract. We prove a stability theorem for frames. Our result is a generalization of a classical result of Paley and Wiener about Riesz bases; it is also related to the Perturbation Theorem of Kato.

The classical Paley-Wiener Theorem states the following: Let \( \{f_i\}_{i=1}^{\infty} \) be a basis for the Banach space \( B \), and let \( \{g_i\}_{i=1}^{\infty} \) be a family of vectors in \( B \). If there exists a constant \( \lambda \in [0, 1] \) such that

\[
\left\| \sum_{i=1}^{n} c_i (f_i - g_i) \right\| \leq \lambda \cdot \left\| \sum_{i=1}^{n} c_i f_i \right\|
\]

for all scalars \( c_1, \ldots, c_n \) \((n = 1, 2, \ldots)\), then \( \{g_i\}_{i=1}^{\infty} \) is a basis for \( B \).

Intuitively, the statement is that any family which is sufficiently close to a basis (in the sense above) is a basis. The proof is not difficult. The conditions imply that there exists a bounded invertible operator \( T \) such that \( Tf_i = g_i \).

The above formulation is due to Boas (cf. [Y]). The Paley-Wiener Theorem is useful in order to show that a family \( \{g_i\}_{i=1}^{\infty} \) is a Riesz basis for a Hilbert space, so the result is sometimes used in wavelet analysis ([B], [S]). But in many cases the wavelet experts prefer to work with frames instead of Riesz bases; our aim here is to show that a similar result holds for frames, however with a completely different proof.

The needed facts about frames can be found in the paper [C].

Theorem 1. Let \( \mathcal{H} \) be a Hilbert space and \( \{f_i\}_{i=1}^{\infty} \) a frame for \( \mathcal{H} \) with bounds \( A \) and \( B \). Let \( \{g_i\}_{i=1}^{\infty} \) be a family of elements in \( \mathcal{H} \), and suppose that

\[
\exists \lambda, \mu \geq 0 : \lambda + \frac{\mu}{\sqrt{A}} < 1 \quad \text{and} \quad \left\| \sum_{i=1}^{n} c_i (f_i - g_i) \right\| \leq \lambda \cdot \left\| \sum_{i=1}^{n} c_i f_i \right\| + \mu \cdot \left[ \sum_{i=1}^{n} |c_i|^2 \right]^{1/2}
\]

for all \( c_1, \ldots, c_n \) \((n = 1, 2, \ldots)\). Then \( \{g_i\}_{i=1}^{\infty} \) is a frame with bounds \( A(1 - (\lambda + \mu/\sqrt{A}))^2 \), \( B(1 + \lambda + \mu/\sqrt{B})^2 \).

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Proof. Consider the operator

\[ U : l^2(N) \rightarrow \mathcal{H}, \quad U\{c_i\} := \sum_{i=1}^{\infty} c_i f_i. \]

Frame theory says that \( U \) is well defined, bounded, and that \( \|U\| \leq \sqrt{B}. \) The assumptions imply that we can define an operator

\[ T : l^2(N) \rightarrow \mathcal{H}, \quad T\{c_i\} := \sum_{i=1}^{\infty} c_i g_i \]

and that

\[ \|U\{c_i\} - T\{c_i\}\| \leq \lambda \cdot \|U\{c_i\}\| + \mu \cdot \|\{c_i\}\|, \quad \forall \{c_i\} \in l^2(N). \]

Therefore

\[ \|T\{c_i\}\| \leq [(1 + \lambda)\sqrt{B} + \mu] \cdot \|\{c_i\}\|, \quad \forall \{c_i\} \in l^2(N). \]

That is, \( \{g_i\}_{i=1}^{\infty} \) is a Bessel sequence with upper bound \( \sqrt{B}(\lambda + 1) + \mu \|^2 = B(1 + \lambda + \mu/\sqrt{B})^2. \)

Now we verify the existence of the lower frame bound for \( \{g_i\}_{i=1}^{\infty}. \) Observe that \( UU^* \) is the frame operator for \( \{f_i\}_{i=1}^{\infty} \) and therefore invertible. Let us consider the operator

\[ U^\dagger : \mathcal{H} \rightarrow l^2(N), \quad U^\dagger f := U^* (UU^*)^{-1} f = \{f, (UU^*)^{-1} f_i\}. \]

\( \{(UU^*)^{-1} f_i\}_{i=1}^{\infty} \) is a frame with upper bound \( 1/A, \) so

\[ \|U^\dagger f\|^2 = \sum_{i=1}^{\infty} \left| \left\langle f, (UU^*)^{-1} f_i \right\rangle \right|^2 \leq \frac{1}{A} \cdot \|f\|^2, \quad \forall f \in \mathcal{H}. \]

Using (1) with \( \{c_i\} = U^\dagger f \) we get

\[ \|f - TU^\dagger f\| \leq \left( \lambda + \frac{\mu}{\sqrt{A}} \right) \cdot \|f\|, \quad \forall f \in \mathcal{H}. \]

Therefore \( TU^\dagger \) is invertible, and

\[ \|TU^\dagger\| \leq 1 + \lambda + \frac{\mu}{\sqrt{A}}, \quad \|T(U^\dagger)^{-1}\| \leq 1/(1 - (\lambda + \mu/\sqrt{A})). \]

Any \( f \in \mathcal{H} \) can be written as

\[ f = TU^\dagger(TU^\dagger)^{-1} f = \sum_{i=1}^{\infty} ((TU^\dagger)^{-1} f_i)(UU^*)^{-1} f_i) g_i; \]

thus

\[ \|f\|^4 = (f, f)^2 = \left| \sum_{i=1}^{\infty} \left( (TU^\dagger)^{-1} f_i, (UU^*)^{-1} f_i \right) (g_i, f) \right|^2 \]

\[ \leq \sum_{i=1}^{\infty} |(TU^\dagger)^{-1} f_i, (UU^*)^{-1} f_i| |(g_i, f)|^2 \]

\[ \leq \frac{1}{A} \cdot \|(TU^\dagger)^{-1} f\|^2 \cdot \sum_{i=1}^{\infty} |(g_i, f)|^2 \]

\[ \leq \frac{1}{A(1 - (\lambda + \mu/\sqrt{A}))^2} \cdot \|f\|^2 \cdot \sum_{i=1}^{\infty} |(g_i, f)|^2, \quad \forall f \in \mathcal{H}. \]
Remarks. (1) Suppose that \( \{f_i\}_{i=1}^{\infty} \) is a frame with bounds \( A, B \) and that \( \{g_i\}_{i=1}^{\infty} \) is any family such that \( R := \sum_{i=1}^{\infty} ||f_i - g_i||^2 < A \). Then the condition in Theorem 1 is satisfied with \( \lambda = 0 \) and \( \mu = \sqrt{R} \). Thus Theorem 1 generalizes Theorem 1 in [C].

(2) The condition in Theorem 1 implies that \( \lambda < 1 \). This is essential. Let \( \{f_i\}_{i=1}^{\infty} \) be an orthonormal basis for \( \mathcal{H} \), and define \( g_i := f_i + f_{i+1} \). Then \( \text{span}\{g_i\} = \mathcal{H} \), but \( \{g_i\} \) is not a frame;

\[
\sum_{i=1}^{n} |\langle f, g_i \rangle|^2 \leq 1/n \cdot \|f\|^2
\]

with \( f := \sum_{i=1}^{n}(-1)^{i-1}e_j \), so the lower frame condition is not satisfied. Since

\[
\sum_{i=1}^{n} c_i(f_i - g_i) = \sum_{i=1}^{n} c_i f_{i+1} = \sum_{i=1}^{n} c_i f_i,
\]

the example corresponds to \( \lambda = 1, \mu = 0 \).

(3) Our result is connected with the work of Kato (e.g., [K], p. 190). Consider the operator \( T \) as a perturbation of \( U \); in the terminology of Kato, the condition in Theorem 1 implies that the "perturbation operator" \( T - U \) is \( U \)-bounded with \( U \)-bound smaller than 1.

(4) In the classical Paley-Wiener Theorem, the conditions imply that

\[
\sum_{i=1}^{n} c_i g_i = 0 \text{ if and only if } \sum_{i=1}^{n} c_i f_i = 0.
\]

So the sets \( \{f_i\} \) and \( \{g_i\} \) must have the same linear dependence. This is automatically satisfied if \( \{f_i\} \) and \( \{g_i\} \) are bases, but in general it is a strong condition. We have avoided this obstacle in Theorem 1; from this point of view the introduction of \( \mu \) plays an important role.

(5) Theorem 1 has applications to the important coherent frames, shortly discussed in [C]. For example, the proof of Theorem 5 in [S] uses the Paley-Wiener Theorem; it can be expected that a similar result can be proved for frames, using Theorem 1. Also, our result is applicable to the important problem of perturbation of the mother wavelet \( f \) in a coherent frame \( \{\pi(x_i)f\}_{i=1}^{\infty} \). We refer to the paper [FZ], where the reader also finds other applications of Theorem 1.

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References


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